Spring 2003 Math 152 03/20/03

Common Exam 2 B. Solutions.

PART 1. Multiple Choice (50 points)

Each problem is worth 5 points. Calculators are not allowed on this part of the exam.

1. $\displaystyle\int \frac{x^3}{x-1}\,dx = \int x^2 + x + 1 + \frac{1}{x+1}\,dx = \frac{x^3}{3} + \frac{x^2}{2} + x + \ln|x+1| + C$
   Answer. C

2. Which of the following integrals is an improper integral?
   Answer. D

3. $\displaystyle\int_3^\infty \frac{1}{x(\ln x)^3}\,dx = -\frac{1}{2(\ln x)^2}\Big|_2^\infty = \frac{1}{2(\ln 3)^2}$
   Answer. B

4. The masses $m_1 = 2$ and $m_2 = 3$ are located at the points $P_1(3,1)$ and $P_2(-1,1)$ respectively.
   Since $\bar{x} = \dfrac{2(3)+3(-1)}{2+3} = \dfrac{3}{5}$ and $\bar{y} = \dfrac{2(1)+3(1)}{2+3} = 1$, the center of mass of the system is $(\frac{3}{5}, 1)$.
   Answer. A

5. The curve $y = x^2 + 1$, $-1 \le x \le 1$ is revolved about the $x-$axis. The surface area obtained is given by the integral

$$\int 2\pi y\,ds = \int_{-1}^{1} 2\pi(x^2+1)\sqrt{1+4x^2}\,dx$$

   Answer. E

6. The velocity $v(t)$ (in feet per second) of a moving object is given below at one second increments. Use Simpson's Rule with $n = 4$ to approximate the total distance travelled from $t = 0$ to $t = 4$ seconds.

t	0	1	2	3	4
v(t)	3	1	3	-1	0

The distance travelled is equal to

$$\int_0^4 v(t)\,dt \approx \frac{\Delta x}{3}(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+f(x_4)) = \frac{1}{3}(3+4(1)+2(3)+4(-1)+0) = 3$$

Answer. A.

7. The length of the parametric curve $x = \cos 2t$, $y = \sin 2t$, $\frac{\pi}{4} \le t \le \frac{\pi}{3}$ is equal to $\int_{\pi/4}^{\pi/3} \sqrt{(2\sin 2t)^2 + (2\cos 2t)^2}\, dt = \int_{\pi/4}^{\pi/3} 2\, dt = \frac{\pi}{6}$
Answer. C

8. Which of the following is the direction field for the equation $\frac{dy}{dx} = \frac{y}{x}$?
Answer. B

9. Consider the integral $\int_0^2 e^{x^2}\, dx$. There is no elementary antiderivative, so the integral is approximated.
Which of the approximate integration formulas underestimates the integral value? (Hint: you may find the graph of $y = e^{x^2}$ helpful)
Answer. C

10. A curve passes through the point $(0, 1)$ and has the slope at any point (x, y) equal to y^2. Then it also passes through the point
A) $(2, \frac{1}{3})$ B) $(2, e)$ C) $(2, \frac{1}{2})$ D) $(2, -1)$ E) $(2, 1)$

Solution. The curve is described by the initial value problem $\frac{dy}{dx} = y^2$, $y(0) = 1$. Separate the variables and integrate to get

$$\frac{dy}{y^2} = dx, \quad \frac{-1}{y} = x + C$$

Use the initial condition $y(0) = 1$ to obtain $C = -1$. Thus, $\frac{-1}{y} = x - 1$, $y = \frac{-1}{x-1}$, and hence, $y(2) = -1$.
Answer. D.

PART 2. Work-Out (50 points)

11. Consider the region bounded by
$y = \cos 2x$, $y = 0$, $x = 0$, $x = \frac{\pi}{4}$.

a) The area of the region$=A = \int_0^{\pi/4} \cos 2x\, dx = \frac{1}{2}$.
b) $\bar{x} = \frac{1}{A}\int_0^{\pi/4} x\cos 2x\, dx = \frac{\pi}{4} - \frac{1}{2}$
(use integration by parts with $u = x$, $dv = cos2x\, dx, du = dx$, $v = \frac{1}{2}\sin 2x$).
c) $\bar{y} = \frac{1}{A}\int_0^{\pi/4} \frac{1}{2}(\cos 2x)^2\, dx = \pi/8$
(use the power reduction formula=half angle formula $\cos 2x = \frac{1+\cos 4x}{2}$

12. A plate of a triangular shape is submerged vertically in water so that its base lies at the surface of the water. The plate's base is 6 meters, its height is 2 meters.
Find the hydrostatic force on one side of the plate. (Use the facts that the density of

water is $\rho = 1000$ kg/m^3 and that the acceleration due to gravity is $g = 9.8$ m/s^2.)
Solution. Consider the $x-$axis going down, with the origin at the surface of water. Then the depth of the slice is x, the length of the slice is $y = \frac{6}{2}(2-x) = 3(2-x)$ (use the similarity of the triangles), the area is $y = 3(2-x)\,dx$ and the hydrostatic force is $\int_0^2 \rho\, g\, x\, 3(2-x)\,dx = 4\rho\, g$ N.
Comment: If you use a different coordinate system, your integral may be different, but the answer should be the same.

13. Newton's law of cooling says that the rate of change of a body's temperature is proportional to the difference between the temperature of the body and the temperature of the surrounding air.
A room is kept at 20° C. A boiling pot of soup was brought into the room, and it has cooled from 100° C to 60° C in 10 minutes.
Formulate the initial value problem for $T(t)$, the temperature of the pot at time t. (In other words, write a differential equation and an initial condition satisfied by $T(t)$.)
DO NOT SOLVE THE DIFFERENTIAL EQUATION.

Solution. Since the temperature of the pot is proportional to the difference between the temperature of the pot and the temperature of the surrounding air, the differential equation for $T(t)$ is $\dfrac{dT}{dt} = k(T-20)$ (actually, $k < 0$, so you may wish to write the equation as $\dfrac{dT}{dt} = -k(T-20)$, $k > 0$).
The initial condition is $T(0) = 100$ (the pot is boiling, so the initial temperature is 100°).

14. Consider the function $f(x) = \dfrac{1}{x^4 + x^2}$.

a) The partial fraction decomposition of the function $f(x) = \dfrac{1}{x^4+x^2} = \dfrac{1}{x^2(x^2+1)}$ has the form $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx+D}{x^2+1}$,
where $A = C = 0$, $B = 1$, $D = -1$.

b) $\int \dfrac{1}{x^4+x^2}\,dx = \int \dfrac{1}{x^2} - \dfrac{1}{x^2+1}\,dx = -\dfrac{1}{x} - \arctan x + C$

c) Since $\int_1^\infty \dfrac{1}{x^4+x^2}\,dx = \lim_{t\to\infty}(-\dfrac{1}{t} - \arctan t) - (-1-\frac{\pi}{2}) = (0-\frac{\pi}{2}) - (-1-\frac{\pi}{4}) = 1-\frac{\pi}{4}$,
the integral converges.
You can also use a comparison with $\int_1^\infty \frac{1}{x^4}\,dx$ or with $\int_1^\infty \frac{1}{x^2}\,dx$ to make the conclusion about the convergence. But here you were asked also to compute the value of the integral, so just the Comparison Theorem is not enough.

15. Consider the initial value problem $\dfrac{dy}{dx} - \dfrac{1}{x}y = x\,e^{-x}$, $y(1) = -\dfrac{1}{e}$.

a) An integrating factor is $\mu(x) = e^{\int p(x)\, dx} = e^{-\ln x} = \frac{1}{x}$.

b) Multiplying both sides by $\mu(x) = \frac{1}{x}$ and integrating $(\mu y)' = e^{-x}$, we get

$$\frac{1}{x}y = -e^{-x} + C,\ y = -xe^{-x} + Cx$$

This is the general solution of the equation.

c) The particular solution satisfying the given initial condition, $y(1) = -\frac{1}{e}$, is $y = -xe^{-x}$, since $C = 0$.

d) $\lim_{x\to\infty} -xe^{-x} = \lim_{x\to\infty} -\frac{x}{e^x} = 0$ (use L'Hospital's Rule).