1. Compute \( \int_1^\infty \frac{1}{\sqrt{x} + x\sqrt{x}} \, dx \) by hand using \( w = \sqrt{x} \).

(\text{Check answer with calculator.})

- Let \( w = \sqrt{x} \). Then \( w^2 = x \) and \( 2w \, dw = dx \).

When \( x = 1 \), \( w = 1 \); as \( x \to \infty \), \( w \to \infty \).

Substitution yields

\[
\int_1^\infty \frac{2w}{w + w^2} \, dw = \int_1^\infty \frac{2}{1+w^2} \, dw = \lim_{b \to \infty} 2 \tan^{-1}(w) \bigg|_1^b = \pi - \frac{\pi}{2} = \frac{\pi}{2} \approx 1.57.
\]

2. Use the Comparison Theorem to determine whether the integral \( \int_0^\infty \frac{\sin^2 x}{\sqrt{x}} \, dx \) is convergent or divergent.

Show your work by hand; check on your calculator.

- Note that \( 0 \leq \frac{\sin^2 x}{\sqrt{x}} \leq \frac{1}{\sqrt{x}} \) for \( 0 < x \leq \pi \).

Moreover, \( \lim_{x \to 0^+} \frac{\sin^2 x}{\sqrt{x}} = 0 \). Since the integral

\[
\int_0^\pi \frac{1}{\sqrt{x}} \, dx = 2\sqrt{x} \approx 3.54 \quad \text{(see below)}
\]

is \( \int_0^\infty \frac{\sin^2 x}{\sqrt{x}} \, dx \) by a comparison theorem. Indeed,

\[
\int_0^\infty \frac{\sin^2 x}{\sqrt{x}} \, dx \approx 1.34.
\]

- Here are details for the comparison integral.

\[
\int_0^\pi \frac{1}{\sqrt{x}} \, dx = \lim_{a \to 0^+} \int_a^\pi x^{-1/2} \, dx = \lim_{a \to 0^+} 2\sqrt{x} \bigg|_a^\pi = 2\sqrt{\pi} - 0 = 2\sqrt{\pi}
\]

3. Solve the initial value problem \( \frac{dy}{dx} = 2x\sqrt{1-y^2}, y(0) = 0 \), explicitly for \( y \) by hand. (Check with calc.)

- The differential equation is separable. Proceed in the usual manner.

\[
\frac{1}{\sqrt{1-y^2}} \, dy = 2xdx
\]

\[
\sin^{-1} y = x^2 + C
\]

\[
0 = C \quad \text{(Sub IC.)}
\]

\[
\sin^{-1} y = x^2
\]

\[
y = \sin(x^2)
\]

4. Solve the differential equation \( x \ln x \frac{dy}{dx} + y = xe^x \) by hand. (Check with your calculator.)

- The differential equation is linear. Its standard form is \( y' + \frac{1}{x} \ln x \, y = e^x \). An integrating factor is \( \mu = \exp \left( \int \frac{1}{x \ln x} \, dx \right) = e^{\ln(x)} = \ln x \).

- Multiply the SLF by \( \mu \): \( y' \ln x + \frac{1}{x} y = e^x \) or \( (y \ln x)' = e^x \). Antidifferentiate and isolate.

\[
y \ln x = e^x + C \implies y = e^x + C.
\]

5. Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings.

A bottle of soda pop at room temperature (72°F) is placed in a refrigerator where the temperature is 44°F. After 30 minutes the soda pop has cooled to 61°F.

(a) Let \( q \) be the temperature of the soda pop. State a differential equation and initial condition that models the description given. Then solve for \( q \) in terms of \( t \) (time). Use deSolve.

- We have \( q' = k(q - 44), q(0) = 72 \), whence \( q = 28e^{kt} + 44 \).

- When \( t = 30 \), \( q = 61 \). Determine \( k \), then \( q \).

\[
\begin{align*}
61 &= 28e^{30k} + 44 \\
7 &= 28e^{30k} \\
\ln \frac{7}{28} &= 30k \\
k &= \frac{1}{30} \ln \frac{7}{28} \\
q &= 28 \left( \frac{7}{28} \right)^{t/30} + 44
\end{align*}
\]

(b) When will the soda cool to 50°F?

- Solve \( 50 = 28 \left( \frac{7}{28} \right)^{t/30} + 44 \) to obtain

\[
t = \frac{30 \ln(14/3)}{\ln(28/17)} \approx 92.61 \text{ minutes.}
\]

6. An epidemic spreads at a rate jointly proportional to the number of infected people and the number of uninfected people. In an isolated town of 5000 inhabitants, 160 people have the disease at the beginning of the week and 1200 have it at the end of the week. How many days does it take for 80% of the population to become infected? Use deSolve.

- Let \( y \) be the number of people infected. Solve \( y' = ky(5000 - y), y(0) = 160 \), to obtain

\[
y = \frac{20000e^{20000t}}{4e^{20000t} + 121}
\]

- When \( t = 7, y = 1200 \). So \( 1200 = \frac{20000e^{20000(7)}}{4e^{20000(7)} + 121} \)

yield \( k = \frac{\ln(363 - \ln(38)}{35000} \). Thus \( y = \frac{20000(363)^{t/7}}{4(363)^{t/7} + 121(38)^{t/7}} \).

- If 80% are infected,

\[
y = 0.80(5000) = 4000 = \frac{20000(363)^{t/7}}{4(363)^{t/7} + 121(38)^{t/7}},
\]

so \( t = \frac{14 \ln(11)}{\ln(363/38)} \approx 14.88 \) or roughly 15 days.
7. Find the arc length of the curve
\[ x = (y - 4)^{3/2}, \quad 5 \leq y \leq 8. \]
• Now \( \frac{dx}{dy} = \frac{3}{2} (y - 4)^{-1/2} \). Therefore,
\[ L = \int_5^8 \sqrt{1 + \left( \frac{3}{2} (y - 4)^{-1/2} \right)^2} \, dy = \frac{80 \sqrt{113} - 13 \sqrt{13}}{27}, \]
approximately 7.63 cm.

8. Find the arc length of the parametric curve
\[ x = 1 + 3t^2, \quad y = 4 + 2t^3, \quad 0 \leq t \leq 1. \]
• Now \( \frac{dx}{dt} = 6t \) and \( \frac{dy}{dt} = 6t^2 \). Therefore,
\[ L = \int_a^b \sqrt{ \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 } \, dt \]
\[ = \int_0^1 6t \sqrt{t^2 + 1} \, dt = 2 \left( 2 \sqrt{2} - 1 \right) \approx 3.66 \text{ cm}. \]

9. Find the area of the surface obtained by rotating the curve \( x = 5 - y^2, \quad 0 \leq y \leq \sqrt{2} \), about the x-axis.
• Now \( \frac{dx}{dy} = -2y \). So \( S = \int 2\pi r \, ds = \int 2\pi y \, ds \)
or \( \int_{\sqrt{2}}^0 2\pi y \sqrt{1 + 4y^2} \, dy = \frac{13}{2} \pi \approx 13.61 \text{ cm}^2. \)

10. Find the area of the surface obtained by rotating the parametric curve \( x = 2t^2 + \frac{1}{t}, \quad y = 8\sqrt{t}, \quad 1 \leq t \leq 3, \)
about the x-axis.
• With \( \frac{dx}{dt} = 4t - \frac{1}{t^2} \) and \( \frac{dy}{dt} = \frac{4}{\sqrt{t}} \), we have \( S = \)
\[ \int 2\pi r \, ds = \int 2\pi y \, ds = \int 2\pi y \sqrt{ \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 } \, dt \]
\[ = \int_1^3 2\pi \left( 8\sqrt{t} \right) \left( 4t + \frac{1}{t^2} \right) \, dt = \frac{32}{3} \pi \left( 3 + 103\sqrt{3} \right), \]
or approximately 1215.76 cm$^2$.

11. Find the center of mass of the region bounded by \( y = x^3 - x \) and \( y = x^2 - 1 \). Density \( \delta = k \) is constant.
• Intersections: \( x^3 - x = x^2 - 1 \) implies \( x = \pm 1 \).
• Mass: \( m = \int_{-1}^1 \int_{x^3 - x}^{x^2 - 1} k \, dy \, dx = \frac{4}{7} \).
• Center of mass:
\[ [\bar{x}, \bar{y}] = \frac{1}{4k/3} \int_{-1}^1 \int_{x^3 - x}^{x^2 - 1} k \cdot [x, y] \, dy \, dx \]
\[ = \left[ -\frac{1}{3}, -\frac{12}{35} \right] \approx \left[ -0.20, -0.34 \right]. \]

12. Masses \( p = [5, 4, 3, 6] \) are located at these points in the plane \( \mathbf{r} = \begin{bmatrix} -4 & 2 \\ 0 & 5 \\ 3 & 2 \\ 1 & -2 \end{bmatrix} \).
(Here \( x \) is column 1 and \( y \) in column 2.) Find the center of mass of the system.
• The total mass is \( m = 5 + 4 + 3 + 6 = 18 \).
• The center of mass is
\[ [\bar{x}, \bar{y}] = \frac{1}{m} \mathbf{p} \mathbf{r} = \left[ -\frac{5}{18}, \frac{4}{3} \right] \approx \left[ -0.28, 1.33 \right]. \]

13. A rectangular tank 8 m long, 4 m wide, and 2 m high contains kerosene (density 820 kg/m$^3$) to a depth of 1.5 m. Find the force on one vertical 4 x 2 meter end.
• The differential area is \( dA = 4 \, dy \).
• The pressure at level \( y \) is \( P = \rho g \left( \frac{3}{2} - y \right) \).
• The differential force is \( dF = P \, dA = 4 \rho g \left( \frac{3}{2} - y \right) \, dy \).
• The total force is \( \int_0^{1.5} 4 \cdot 820 \cdot \frac{9}{10} \left( \frac{3}{2} - y \right) \, dy = 36162 \text{ N}. \)

14. A triangular vertical plate is submerged in water (density \( \rho = 1000 \text{ kg/m}^3 \)) as depicted below. Find the hydrostatic force against it.

15. Determine if the sequence \( a_n = \frac{n^2 \cos n}{1 + n^2} \) converges.
(A sequence plot will help. Sketch it below.)
• The limit \( \lim a_n = \lim \frac{\cos n}{n^2 + 1} \) does not exist.
Hence the sequence diverges by the Test for Divergence.

16. Find the limit of the positive recursive sequence \( a_1 = 1, \quad a_{n+1} = 1 + \frac{1}{a_n} \).
• Call the limit \( p \). Then as \( n \to \infty \), we have \( a_n, a_{n+1} \to p \). This yields
\[ p = 1 + \frac{1}{p} \]
whence \( p = \frac{1 \pm \sqrt{5}}{2} \). Since the sequence is positive, we see that \( p = \frac{1 + \sqrt{5}}{2} \approx 1.62 \), the Golden Ratio!
17. Determine if the series \( \sum_{n=1}^{\infty} \frac{1}{1 + \left( \frac{3}{2} \right)^n} \) converges. 
Cite a test. If it converges, find its sum.

- Note that \( a_n \to 1 \neq 0 \). The series diverges by the Test for Divergence.

18. Express the following number as a ratio of positive integers (like I did in elementary school).

\[ 2.516 = 2.516516516 \ldots \]

- Let \( x \) be the number. Subtract \( x \) from 1000, then divide.

\[
1000x = 2516.516 \\
x = 2.516
\]

So 999x = 2514, whence \( x = \frac{2514}{999} = \frac{838}{333} \).

19. Show that the series converges by applying the Comparison Test or Integral Test (your choice). Compute its 10th partial sum, both exactly and approximately. Obtain an error estimate for the remainder via an improper integral. 

\[ \sum_{n=1}^{\infty} \frac{1}{n^2 + n^3} \]

- Compare terms of this positive series with those of a convergent \( p \)-series: \( \frac{1}{n^{p+1}} < \frac{1}{n^p} \) (\( p = 2 > 1 \)). By the Comparison Test, our series converges.

- Now \( s_{10} = \sum_{n=1}^{10} \frac{1}{n^2 + n^3} \approx 0.64 \).

- The remainder satisfies \( R \leq \int_{10}^{\infty} \frac{1}{x^2 + x^3} \, dx \), which equals \( \frac{1}{10} + \ln 10 - \ln 11 \approx 4.69 \times 10^{-3} \).

20. Determine whether the series \( \sum_{n=1}^{\infty} \sin \left( \frac{1}{n} \right) \) converges or diverges. I recommend the Limit Comparison Test, which is especially tasty in this instance!

- Compare the series with the harmonic series \( \sum \frac{1}{n} \).

- Now \( \lim_{n \to \infty} \frac{\sin \left( \frac{1}{n} \right)}{\frac{1}{n}} = 1 \). Since the harmonic series diverges, so does the series \( \sum \sin \left( \frac{1}{n} \right) \) by the Limit Comparison Test.