MATH 152, FALL 2015
COMMON EXAM III - VERSION A

LAST NAME(print): ___________________________ FIRST NAME(print): ___________________________

INSTRUCTOR: ___________________________

SECTION NUMBER: _______________

DIRECTIONS:

1. The use of a calculator, laptop or computer is prohibited.

2. TURN OFF cell phones and put them away. If a cell phone is seen during the exam, your exam will be collected and you will receive a zero.

3. In Part 1 (Problems 1-15), mark the correct choice on your ScanTron using a No. 2 pencil. The scantrons will not be returned, therefore for your own records, also record your choices on your exam!

4. In Part 2 (Problems 16-20), present your solutions in the space provided. Show all your work neatly and concisely and clearly indicate your final answer. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

5. Be sure to write your name, section number and version letter of the exam on the ScanTron form.

THE AGGIE CODE OF HONOR

"An Aggie does not lie, cheat or steal, or tolerate those who do."

Signature: ___________________________

1
PART I: Multiple Choice. 4 points each

1. The series \( \sum_{n=1}^{\infty} \frac{\sin n + 5}{n^{3/2}} \)
   (a) Converges by the Comparison Test with \( \sum_{n=1}^{\infty} \frac{6}{n^{3/2}} \)
   (b) Converges by the Comparison Test with \( \sum_{n=1}^{\infty} \frac{4}{n^{3/2}} \)
   (c) Diverges by the Comparison Test with \( \sum_{n=1}^{\infty} \frac{6}{n^{3/2}} \)
   (d) Diverges by the Comparison Test with \( \sum_{n=1}^{\infty} \frac{4}{n^{3/2}} \)
   (e) Diverges by the Test for Divergence

\[
\frac{\sin n + 5}{n^{3/2}} \leq \frac{6}{n^{3/2}}
\]
\( \sum_{n=1}^{\infty} \frac{6}{n^{3/2}} \) is a multiple of a convergent p-series and converges.
So, by Comparison Test,
\( \sum_{n=1}^{\infty} \frac{\sin n + 5}{n^{3/2}} \) also converges.

2. What is the intersection of the sphere \((x - 2)^2 + (y - 3)^2 + (z + 2)^2 = 17\) with the yz-plane?
   (a) A circle centered at \((3, -3, 2)\) with radius \(\sqrt{13}\).
   (b) A circle centered at \((2, 0, 0)\) with radius 2.
   (c) A circle centered at \((0, 3, -2)\) with radius \(\sqrt{13}\).
   (d) A circle centered at \((-2, 0, 0)\) with radius 2.
   (e) The points \((4, 0, 0)\) and \((0, 0, 0)\).

\[
(2)^2 + (y - 3)^2 + (z + 2)^2 = 17
\]
\[
(y - 3)^2 + (z + 2)^2 = 13
\]

3. Find the Maclaurin series for the function \( f(x) = x^2 e^{-x^3} \)
   (a) \( \sum_{n=0}^{\infty} \frac{x^{3n+2}}{n!} \)
   (b) \( \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+2}}{n!} \)
   (c) \( \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+6}}{n!} \)
   (d) \( \sum_{n=0}^{\infty} \frac{x^{3n+6}}{n!} \)
   (e) \( \sum_{n=0}^{\infty} \frac{x^{5n}}{n!} \)

\[

x^2 \sum_{n=0}^{\infty} \frac{(x^3)^n}{n!} = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!}
\]

4. The alternating series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{(n + 2)^2} \) converges. Use the Alternating Series Estimation Theorem to determine an upper bound on the absolute value of the error in using \( s_6 \) to approximate the sum of the series.
   (a) \( \frac{1}{64} \)
   (b) \( \frac{1}{64} \)
   (c) \( \frac{1}{36} \)
   (d) \( \frac{1}{8} \)
   (e) \( \frac{1}{9} \)

\[
|R_6| \leq b_7 = \frac{1}{9^2} = \frac{1}{81}
\]
5. Which of the following is true regarding the series \( \sum_{n=1}^{\infty} \frac{5n \cdot 3^{2n}}{4^n} \)?

(a) The Ratio Test limit is \( \frac{3}{4} \), so the series diverges.
(b) The Ratio Test limit is \( \frac{3}{4} \), so the series converges.
(c) The Ratio Test limit is \( \frac{15}{4} \), so the series diverges.
(d) The Ratio Test limit is \( \frac{9}{4} \), so the series diverges.
(e) The Ratio Test limit is \( \frac{9}{4} \), so the series converges.

\(\lim_{n \to \infty} \left| \frac{5(n+1)3^{2(n+1)}}{4^{n+1}} \cdot \frac{4^n}{5n \cdot 3^{2n}} \right| = \frac{5 \cdot 3^2}{4 \cdot 5} = \frac{9}{4} > 1\)

6. Consider the series below. Which statement is true regarding the absolute convergence of each series?

(I) \( \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2 + 3} \)  
(II) \( \sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n + 1} \)

(a) I converges absolutely, II converges but not absolutely
(b) Both I and II converge but not absolutely
(c) Both I and II converge absolutely
(d) I diverges, II converges absolutely
(e) I converges but no absolutely, II converges absolutely

7. The \( p \)-series \( \sum_{n=1}^{\infty} \frac{1}{n^3} \) converges by the Integral Test. How many terms are needed to guarantee that the \( n \)th partial sum \( s_n \) is accurate to within \( \frac{1}{100} \)?

(a) 5 terms
(b) 6 terms
(c) 7 terms
(d) 8 terms
(e) 9 terms

\[ \frac{1}{2n^2} \leq \frac{1}{100} \]

\[ 2n^2 \geq 100 \]

\[ n^2 \geq 50 \]

\[ n \geq \sqrt{50} \]

8. Find the sum of the series \( \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1}}{4^{2n}(2n)!} \)

(a) \( 3 \cos \left( \frac{3}{4} \right) \)
(b) \( 3 \sin \left( \frac{3}{4} \right) \)
(c) \( \cos \left( \frac{3}{4} \right) \)
(d) \( \sin \left( \frac{3}{4} \right) \)
(e) \( 3 \sin \left( -\frac{3}{4} \right) \)

\[ \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1}}{4^{2n}(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n (3/4)^{2n}}{(2n)!} = 3 \cos \left( \frac{3}{4} \right) \]
9. The series \( \sum_{n=2}^{\infty} c_n x^n \) converges when \( x = 4 \) and diverges when \( x = -7 \). What can be said about the convergence of the following series?

\[
\begin{align*}
(\text{I}) & \quad \sum_{n=2}^{\infty} c_n 9^n \\
(\text{II}) & \quad \sum_{n=2}^{\infty} c_n (-4)^n
\end{align*}
\]

(a) Both I and II are inconclusive
(b) I diverges, II converges
(c) I diverges, II is inconclusive
(d) Both I and II converge
(e) I is inconclusive, II converges

10. Find the radius and interval of convergence of the series \( \sum_{n=0}^{\infty} \frac{n! (x + 4)^n}{3^n} \).

(a) \( R = 0 \); No interval of convergence since the series never converges.
(b) \( R = 0 \), \( I = \{-4\} \)
(c) \( R = \infty \), \( I = (-\infty, \infty) \)
(d) \( R = 3 \), \( I = (-7, -1) \)
(e) \( R = \infty \), \( I = \{-4\} \)

11. Find the Maclaurin series for the function \( f(x) = \frac{x^3}{1 - 5x} \).

(a) \( \sum_{n=0}^{\infty} (-1)^n 5^n x^{n+3} \)
(b) \( \sum_{n=0}^{\infty} 5^n x^{3n} \)
(c) \( \sum_{n=0}^{\infty} (-1)^n 5^n x^{3n} \)
(d) \( \sum_{n=0}^{\infty} (-1)^n 5^n x^{n+3} \)
(e) \( \sum_{n=0}^{\infty} 5^n x^{n+3} \)

12. Find the radius of the sphere \( x^2 - 2x + y^2 + 6y + z^2 = 13 \).

(a) 23
(b) 3
(c) \( \sqrt{3} \)
(d) \( \sqrt{23} \)
(e) \( \sqrt{17} \)

\[
\begin{align*}
\text{X}^2 - 2x + 1 + y^2 + 6y + 9 + z^2 &= 13 + 1 + 9 \\
(x-1)^2 + (y+3)^2 + z^2 &= 23 \\
r^2 &= 23 \\
r &= \sqrt{23}
\end{align*}
\]
13. Find the 3rd degree Taylor polynomial, \( T_3(x) \), for the function \( f(x) = x^{3/2} \) centered at \( a = 1 \).

(a) \( T_3(x) = 1 + \frac{3}{2} (x - 1) + \frac{3}{4} (x - 1)^2 - \frac{3}{8} (x - 1)^3 \)

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<td>( -\frac{3}{8} x^{-3/2} )</td>
<td>( -\frac{3}{8} )</td>
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\[ T_3(x) = 1 + \frac{3}{2} (x - 1) + \frac{3}{4} (x - 1)^2 - \frac{3}{8} (x - 1)^3 \]

14. Evaluate the indefinite integral \( \int \arctan(4x^3) \, dx \) as a Maclaurin series.

(a) \( \sum_{n=0}^{\infty} (\frac{(-1)^n}{2n+1}(4x^3)^{2n+1}) + C \)

\[ \arctan(4x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \]

\[ \int \arctan(4x^3) \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(6n+4)} + C \]

15. Consider the function \( f(x) = e^{-2x} \). Use Taylor's Inequality to give an upper bound on the remainder when using \( T_2(x) \) centered at \( a = 3 \) to approximate this function on the interval \( 1 \leq x \leq 4 \).

Taylor's Inequality: \( |R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1} \) where \( |f^{(n+1)}(x)| \leq M \) for \( 1 \leq x \leq 4 \)

(a) \( \frac{4}{3} e^{-8} \)
(b) \( \frac{92}{3} e^{-8} \)
(c) \( \frac{4}{3} e^{-2} \)
(d) \( \frac{92}{3} e^{-2} \)
(e) \( 36e^{-2} \)

\[ |R_2(3)| \leq \frac{8e^{-2}}{3!} (2)^3 = \frac{64e^{-2}}{6} = \frac{32e^{-2}}{3} \]
PART II WORK OUT

Directions: Present your solutions in the space provided. Show all your work neatly and concisely and box your final answer. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

16. (8 points) Consider the series \( \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n} \).

(a) Show that this series converges.

Alternating Series Test:

1. \( \lim_{n \to \infty} \frac{1}{n \ln n} = 0 \) \( \checkmark \)

2. Decreasing? Since \( \frac{1}{n \ln n} \leq \frac{1}{(n+1) \ln (n+1)} \) for all \( n \geq 2 \),
   then \( b_n \geq b_{n+1} \) for all \( n \geq 2 \).

By AST, series converges.

(b) Determine whether this series converges absolutely.

Consider \( \sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n \ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{n \ln n} \)

Let \( f(x) = \frac{1}{x \ln x} \). \( f \) is continuous, positive, and decreasing for all \( x \geq 2 \).

\[
\int_{2}^{\infty} \frac{1}{x \ln x} \, dx = \lim_{b \to \infty} \int_{2}^{b} \frac{1}{x \ln x} \, dx = \lim_{b \to \infty} \left[ \ln |\ln x| \right]_{2}^{b} = \lim_{b \to \infty} \left[ \ln (\ln b) - \ln (\ln 2) \right] = \infty
\]

Since the improper integral diverges, so does the series by the Integral Test.

Therefore \( \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n} \) does NOT absolutely converge.
17. (7 points) Find the Taylor series for the function \( f(x) = \frac{1}{7x+1} \) centered at \( a = 1 \). Express your answer in summation notation.

\[
\begin{align*}
| & n & f^{(n)}(x) & f^{(n)}(1) \\
| & 0 & \frac{1}{7x+1} & \frac{1}{8} \\
| & 1 & -\frac{1.7}{(7x+1)^2} & -\frac{1.7}{8^2} \\
| & 2 & \frac{2\cdot1.7^2}{(7x+1)^3} & \frac{2\cdot1.7^2}{8^3} \\
| & 3 & -\frac{3\cdot2\cdot1.7^3}{(7x+1)^4} & -\frac{3\cdot2\cdot1.7^3}{8^4} \\
\end{align*}
\]

\[f^{(n)}(1) = \frac{(-1)^n n! 7^n}{8^{n+1}}\]

So,
\[f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!} (x-1)^n = \sum_{n=0}^{\infty} \frac{(-1)^n n! 7^n}{8^{n+1}} (x-1)^n\]

18. (5 points) Determine whether the series \( \sum_{n=2}^{\infty} \frac{\sqrt{n+2}}{n^3-n} \) converges or diverges.

Consider \( \sum_{n=2}^{\infty} \frac{\sqrt{n}}{n^3-n} \), a convergent p-series.

\[\lim_{n \to \infty} \frac{\sqrt{n+2}}{n^3-n} = \lim_{n \to \infty} \frac{(\sqrt{n+2})n^{3/2}}{n^3-n} = \lim_{n \to \infty} \frac{n^{3/2}+2n^{3/2}}{n^3-n} = 1 > 0\]

By Limit Comparison Test, both series do the same thing.

Therefore, \( \sum_{n=2}^{\infty} \frac{\sqrt{n+2}}{n^3-n} \) also converges.
19. (12 points)

(a) Find the Maclaurin series for the function \( f(x) = \frac{1}{9 + x^2} \).

\[
\frac{1}{9 + x^2} = \frac{1}{9} \cdot \frac{1}{1 + \frac{x^2}{9}} = \frac{1}{9} \cdot \frac{1}{1 - \left(-\frac{x^2}{9}\right)} = \frac{1}{9} \sum_{n=0}^{\infty} \frac{(-x^2)^n}{9^n}
\]

\[
= \frac{1}{9} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{q^n} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{q^{n+1}}
\]

(b) What is the radius of convergence for this series?

Need \( \left| \frac{-x^2}{9} \right| < 1 \) for a geometric series to converge.

\( |x^2| < 9 \)
\( |x| < 3 \)

\[ R = 3 \]

(c) Use part (a) to find the Maclaurin series for the function \( f(x) = \ln(9 + x^2) \).

\[
\frac{d}{dx} \ln(9 + x^2) = \frac{2x}{9 + x^2} = 2x \cdot \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{q^{n+1}} = \sum_{n=0}^{\infty} \frac{2(-1)^n x^{2n+1}}{q^{n+1}}
\]

So, \( \ln(9 + x^2) = \int \sum_{n=0}^{\infty} \frac{2(-1)^n x^{2n+1}}{q^{n+1}} \, dx \)

\[
= \sum_{n=0}^{\infty} \frac{2(-1)^n x^{2n+2}}{q^{n+1}(2n+2)} + C
\]

When \( x = 0 \): \( \ln 9 = \sum_{n=0}^{\infty} + C \rightarrow C = \ln 9 \)

\[
\ln(9 + x^2) = \sum_{n=0}^{\infty} \frac{2(-1)^n x^{2n+2}}{q^{n+1}(2n+2)} + \ln 9
\]
20. (8 points) Find the radius and interval of convergence of the power series \( \sum_{n=2}^{\infty} \frac{(-1)^n(3x - 1)^n}{5^n(n-1)} \).

**Ratio Test:**

\[
\lim_{n \to \infty} \left| \frac{(3x-1)^n}{5^n(n-1)} \cdot \frac{5^n(n-1)}{(3x-1)^n} \right| = \lim_{n \to \infty} \left| \frac{5^n(n-1)}{5^n(n-1)} \right| = 1.
\]

\[
\left| \frac{3x-1}{5} \right| < 1, \text{ so } |3x-1| < 5
\]

\[
\left| x - \frac{1}{3} \right| < \frac{5}{3}
\]

Center: \( \frac{1}{3} \); \( R = \frac{5}{3} \)

\[x = -\frac{4}{3} : \sum_{n=2}^{\infty} \frac{(-1)^n(-6)^n}{5^n(n-1)} = \sum_{n=2}^{\infty} \frac{(-1)^n(-1)^n5^n}{5^n(n-1)}
\]

\[= \sum_{n=2}^{\infty} \frac{1}{n-1} \geq \sum_{n=2}^{\infty} \frac{1}{n} \text{. By Comparison with the divergent harmonic series, } \sum_{n=2}^{\infty} \frac{1}{n-1} \text{ diverges.}
\]

\[x = 2 : \sum_{n=2}^{\infty} \frac{(-1)^n5^n}{5^n(n-1)} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n-1} \text{ converges by AST.}
\]

**Interval:** \( \left[ -\frac{4}{3}, 2 \right] \)

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