MATH 152, FALL 2015
COMMON EXAM III - VERSION B

LAST NAME(print): ___________________________ FIRST NAME(print): ___________________________

INSTRUCTOR: ___________________________

SECTION NUMBER: ___________________________

DIRECTIONS:

1. The use of a calculator, laptop or computer is prohibited.

2. TURN OFF cell phones and put them away. If a cell phone is seen during the exam, your exam will be collected and you will receive a zero.

3. In Part 1 (Problems 1-15), mark the correct choice on your ScanTron using a No. 2 pencil. The scantrons will not be returned, therefore for your own records, also record your choices on your exam!

4. In Part 2 (Problems 16-20), present your solutions in the space provided. Show all your work neatly and concisely and clearly indicate your final answer. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

5. Be sure to write your name, section number and version letter of the exam on the ScanTron form.

THE AGGIE CODE OF HONOR

“An Aggie does not lie, cheat or steal, or tolerate those who do.”

Signature: ___________________________
PART I: Multiple Choice. 4 points each

1. Find the radius of the sphere \( x^2 - 2x + y^2 + 6y + z^2 = 13 \).
   \[ (x - 1)^2 + (y + 3)^2 + z^2 = 23 \]
   \( r^2 = 23 \)
   \( r = \sqrt{23} \)
   (c) \( \sqrt{23} \)
   (d) 3
   (e) \( \sqrt{3} \)

2. Find the sum of the series \( \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1}}{4^{2n}(2n)!} = \sum_{n=0}^{\infty} 3 \cdot (-1)^n \left( \frac{3}{4} \right)^{2n} \frac{(2n)!}{(2n)!} = 3 \cos \left( \frac{3}{4} \right) \).
   (a) \( \sin \left( \frac{3}{4} \right) \)
   (b) \( 3 \cos \left( \frac{3}{4} \right) \)
   (c) \( 3 \sin \left( \frac{3}{4} \right) \)
   (d) \( \cos \left( \frac{3}{4} \right) \)
   (e) \( 3 \sin \left( -\frac{3}{4} \right) \)

3. Which of the following is true regarding the series \( \sum_{n=1}^{\infty} \frac{5n \cdot 3^{2n}}{4^n} \).
   \( \lim_{n \to \infty} \left| \frac{5(n+1) 3^{2(n+1)}}{4^{n+1}} \cdot \frac{4^n}{5n \cdot 3^{2n}} \right| = \lim_{n \to \infty} \left| \frac{5(n+5) 3^{2n+2}}{4^n \cdot 4 \cdot 5n \cdot 3^{2n}} \right| = \frac{5 \cdot 9}{4 \cdot 5} = \frac{9}{4} > 1 \)
   (a) The Ratio Test limit is \( \frac{15}{4} \), so the series diverges.
   (b) The Ratio Test limit is \( \frac{9}{4} \), so the series converges.
   (c) The Ratio Test limit is \( \frac{3}{4} \), so the series diverges.
   (d) The Ratio Test limit is \( \frac{3}{4} \), so the series converges.
   (e) The Ratio Test limit is \( \frac{9}{4} \), so the series diverges.

4. Evaluate the indefinite integral \( \int \arctan(4x^3) \, dx \) as a Maclaurin series.
   \[ \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n+1} x^{6n+4}}{(2n+1)(6n+4)} + C \]
   (a) \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n+1} x^{6n+4}}{(2n+1)(6n+4)} + C
   (b) \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n+1} x^{6n}}{(2n+1)(6n+2)} + C
   (c) \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n+1} x^{6n+4}}{(2n+1)(6n+4)} + C
   (d) \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n+1} x^{6n+4}}{(2n+1)(6n+4)} + C
   (e) \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n+1} x^{6n+4}}{(2n+1)(6n+4)} + C
5. Find the Maclaurin series for the function \( f(x) = \frac{x^3}{1 - 5x} \).

(a) \( \sum_{n=0}^{\infty} (-1)^n 5^n x^{3n} \)

(b) \( \sum_{n=0}^{\infty} 5^n x^{n+3} \)

(c) \( \sum_{n=0}^{\infty} (-1)^n 5^n x^{n+3} \)

(d) \( \sum_{n=0}^{\infty} 5^n x^{3n} \)

(e) \( \sum_{n=0}^{\infty} (-1)^n 5^n x^{n+3} \)

\[ f(x) = x^3 \cdot \frac{1}{1 - 5x} = x^3 \sum_{n=0}^{\infty} (5x)^n = x^3 \sum_{n=0}^{\infty} 5^n x^n = \sum_{n=0}^{\infty} 5^n x^{n+3} \]

6. Find the 3rd degree Taylor polynomial, \( T_3(x) \), for the function \( f(x) = x^{3/2} \) centered at \( a = 1 \).

(a) \( T_3(x) = 1 + \frac{3}{2} (x-1) + \frac{3}{4} (x-1)^2 - \frac{3}{8} (x-1)^3 \)

(b) \( T_3(x) = 1 + \frac{3}{2} (x-1) + \frac{3}{4} (x-1)^2 - \frac{3}{8} (x-1)^3 \)

(c) \( T_3(x) = 1 + \frac{3}{2} (x-1) + \frac{3}{4} (x-1)^2 - \frac{1}{16} (x-1)^3 \)

(d) \( T_3(x) = 1 + \frac{3}{2} (x-1) + \frac{3}{4} (x-1)^2 - \frac{1}{16} (x-1)^3 \)

(e) \( T_3(x) = 1 + \frac{3}{2} (x-1) + \frac{3}{4} (x-1)^2 - \frac{1}{16} (x-1)^3 \)

\[ T_3(x) = 1 + \frac{3}{2} (x-1) + \frac{3}{4} (x-1)^2 - \frac{3}{8} (x-1)^3 \]

7. The series \( \sum_{n=1}^{\infty} \frac{\sin n + 5}{n^{3/2}} \)

(a) Diverges by the Comparison Test with \( \sum_{n=1}^{\infty} \frac{4}{n^{3/2}} \)

(b) Converges by the Comparison Test with \( \sum_{n=1}^{\infty} \frac{4}{n^{3/2}} \)

(c) Diverges by the Comparison Test with \( \sum_{n=1}^{\infty} \frac{6}{n^{3/2}} \)

(d) Diverges by the Test for Divergence

(e) Converges by the Comparison Test with \( \sum_{n=1}^{\infty} \frac{6}{n^{3/2}} \)

\[ \sum_{n=1}^{\infty} \frac{6}{n^{3/2}} \] is a multiple of a convergent p-series \( (p = 3/2 > 1) \) Therefore, \( \sum_{n=1}^{\infty} \frac{\sin n + 5}{n^{3/2}} \) converges by Comparison Test.
8. What is the intersection of the sphere \((x - 2)^2 + (y - 3)^2 + (z + 2)^2 = 17\) with the \(yz\)-plane?\\
(a) A circle centered at \((-2, 0, 0)\) with radius 2.\\
(b) A circle centered at \((0, 3, -2)\) with radius \(\sqrt{13}\).\\
(c) The points \((4, 0, 0)\) and \((0, 0, 0)\).\\
(d) A circle centered at \((0, -3, 2)\) with radius \(\sqrt{13}\).\\
(e) A circle centered at \((2, 0, 0)\) with radius 2.

9. Consider the series below. Which statement is true regarding the absolute convergence of each series?\\
\[
(I) \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2 + 3} \quad (II) \sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{4^n + 1}
\]

(a) I converges but not absolutely, II converges absolutely\\
(b) Both I and II converge but not absolutely\\
(c) I converges absolutely, II converges but not absolutely\\
(d) Both I and II converge absolutely\\
(e) I diverges, II converges absolutely

10. Find the radius and interval of convergence of the series \[
\sum_{n=0}^{\infty} \frac{n!(x+4)^n}{3^n}.\]

(a) \(R = \infty, I = (-4, \infty)\)\\
(b) \(R = 0;\) No interval of convergence since the series never converges.\\
(c) \(R = \infty, I = (-\infty, \infty)\)\\
(d) \(R = 4, I = (-8, 0)\)\\
(e) \(R = 3, I = (-7, -1)\)

11. Consider the function \(f(x) = e^{-2x}\). Use Taylor's Inequality to give an upper bound on the remainder when using \(T_2(x)\) centered at \(a = 3\) to approximate this function on the interval \(1 \leq x \leq 4\).

Taylor's Inequality: \(|R_2(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}\) where \(|f^{(n+1)}(x)| \leq M\) for \(1 \leq x \leq 4\).

(a) \(\frac{4}{3}e^2\)\\
(b) \(\frac{8}{3}e^2\)\\
(c) \(\frac{1}{3}e^2\)\\
(d) \(\frac{32}{3}e^8\)\\
(e) \(36e^{-2}\)

\[
|R_2(x)| \leq \frac{8e^{-2}}{3!} (2)^3 = \frac{64e^{-2}}{6} = \frac{32}{3}e^{-2}
\]
12. The alternating series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+2)^2} \) converges. Use the Alternating Series Estimation Theorem to determine an upper bound on the absolute value of the error in using \( s_6 \) to approximate the sum of the series.

(a) \( \frac{1}{8} \)
(b) \( \frac{1}{64} \)
(c) \( \frac{1}{9} \)
(d) \( \frac{1}{35} \)
(e) \( \frac{1}{81} \)

13. Find the Maclaurin series for the function \( f(x) = x^2 e^{-x^3} \).

(a) \( \sum_{n=0}^{\infty} \frac{x^{3n+6}}{n!} \)
(b) \( \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+6}}{n!} \)
(c) \( \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+2}}{n!} \)
(d) \( \sum_{n=0}^{\infty} \frac{x^{5n}}{n!} \)
(e) \( \sum_{n=0}^{\infty} \frac{x^{3n+2}}{n!} \)

14. The \( p \)-series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) converges by the Integral Test. How many terms are needed to guarantee that the \( n \)th partial sum \( s_n \) is accurate to within \( \frac{1}{100} \)?

(a) 8 terms
(b) 5 terms
(c) 9 terms
(d) 6 terms
(e) 7 terms

15. The series \( \sum_{n=2}^{\infty} c_n x^n \) converges when \( x = 4 \) and diverges when \( x = -7 \). What can be said about the convergence of the following series?

(I) \( \sum_{n=2}^{\infty} c_n 9^n \)
(II) \( \sum_{n=2}^{\infty} c_n (-4)^n \)

(a) I diverges, II is inconclusive
(b) I is inconclusive, II converges
(c) Both I and II are inconclusive
(d) I diverges, II converges
(e) Both I and II converge
PART II WORK OUT

Directions: Present your solutions in the space provided. Show all your work neatly and concisely and Box your final answer. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

16. (8 points) Consider the series \( \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n} \).

(a) Show that this series converges.

**Alternating Series Test:**

1. \( \lim_{n \to \infty} \frac{1}{n \ln n} = 0 \) because \( \ln n \) grows slower than \( n \).

2. Decreasing? Since \( \frac{1}{n \ln n} \geq \frac{1}{(n+1) \ln (n+1)} \) for all \( n \geq 2 \), then \( b_n \) is decreasing.

By AST, \( \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n} \) converges.

(b) Determine whether this series converges absolutely.

Consider \( \sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n \ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{n \ln n} \).

Let \( f(x) = \frac{1}{x \ln x} \). \( f \) is continuous, positive, and decreasing for all \( x \geq 2 \).

\[
\int_{2}^{\infty} \frac{1}{x \ln x} \, dx = \lim_{t \to \infty} \int_{2}^{t} \frac{1}{x \ln x} \, dx = \lim_{t \to \infty} \ln |\ln x| \bigg|_{2}^{t} \\
= \lim_{t \to \infty} \ln (\ln t) - \ln (\ln 2) = \infty
\]

Since improper integral diverges, so does the series \( \sum_{n=2}^{\infty} \frac{1}{n \ln n} \) by the Integral Test.

Therefore, \( \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n} \) does \underline{NOT} absolutely converge.
17. (7 points) Find the Taylor series for the function \( f(x) = \frac{1}{6x - 1} \) centered at \( a = 1 \). Express your answer in summation notation.

\[
\begin{array}{c|c|c}
 n & f^{(n)}(x) & f^{(n)}(1) \\
\hline
 0 & \frac{1}{6x-1} & \frac{1}{3} \\
 1 & -\frac{6}{(6x-1)^2} & -\frac{6}{5^2} \\
 2 & \frac{2\cdot6^2}{(6x-1)^3} & \frac{2\cdot6^2}{5^3} \\
 3 & -\frac{3\cdot2\cdot6^3}{(6x-1)^4} & -\frac{3\cdot2\cdot6^3}{5^4} \\
 4 & \frac{4\cdot3\cdot2\cdot6^4}{(6x-1)^5} & \frac{4\cdot3\cdot2\cdot6^4}{5^5} \\
\end{array}
\]

\[
f^{(n)}(1) = \frac{(-1)^n 6^n n!}{5^{n+1}}
\]

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!} (x-1)^n
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n 6^n n!}{n! 5^{n+1}} (x-1)^n
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n 6^n}{5^{n+1}} (x-1)^n
\]

18. (5 points) Determine whether the series \( \sum_{n=2}^{\infty} \frac{\sqrt{n} + 3}{n^4 - n} \) converges or diverges.

Consider the series \( \sum_{n=2}^{\infty} \frac{\sqrt{n}}{n^4 - n} \), a convergent p-series.

\[
\lim_{n \to \infty} \frac{\sqrt{n} + 3}{n^4 - n} = \lim_{n \to \infty} \frac{(\sqrt{n} + 3)n^{7/2}}{n^4 - n} = \lim_{n \to \infty} \frac{n^{4/2} + 3n^{7/2}}{n^4 - n} = 1 > 0
\]

By Limit Comparison Test, both series must do the same thing.

Therefore, \( \sum_{n=2}^{\infty} \frac{\sqrt{n} + 3}{n^4 - n} \) also converges.
19. (12 points)

(a) Find the Maclaurin series for the function \( f(x) = \frac{1}{4 + x^2} \).

\[
\frac{1}{4 + x^2} = \frac{1}{4} \left( 1 + \frac{x^2}{4} \right)^{-1} = \frac{1}{4} \left[ 1 - \left( -\frac{x^2}{4} \right) \right] = \frac{1}{4} \sum_{n=0}^{\infty} \left( \frac{-x^2}{4} \right)^n
\]

\[
= \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^{n+1}}
\]

(b) What is the radius of convergence for this series?

\[
\text{Need } \left| \frac{-x^2}{4} \right| < 1 \text{ for a geometric series to converge.}
\]

\[
|\frac{x^2}{4}| < 1 \quad \Rightarrow \quad |x^2| < 4 \quad \Rightarrow \quad |x| < 2
\]

\[
R = 2
\]

(c) Use part (a) to find the Maclaurin series for the function \( f(x) = \ln(4 + x^2) \).

\[
\frac{d}{dx} \ln(4 + x^2) = \frac{2x}{4 + x^2} = 2x \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^{n+1}} = \sum_{n=0}^{\infty} \frac{2(-1)^n x^{2n+1}}{4^{n+1}}
\]

So, \( \ln(4 + x^2) = \int \left[ \sum_{n=0}^{\infty} \frac{2(-1)^n x^{2n+1}}{4^{n+1}} \right] dx \)

\[
= \sum_{n=0}^{\infty} \frac{2(-1)^n x^{2n+2}}{4^{n+1} (2n+2)} + C
\]

When \( x = 0 \): \( \ln(4) = \sum_{n=0}^{\infty} C \Rightarrow C = \ln 4 \)

\[
\ln(4 + x^2) = \sum_{n=0}^{\infty} \frac{2(-1)^n x^{2n+2}}{4^{n+1} (2n+2)} + \ln 4
\]
20. (8 points) Find the radius and interval of convergence of the power series \( \sum_{n=2}^{\infty} \frac{(-1)^n(4x-1)^n}{9^n(n-1)} \).

**Ratio Test:**
\[
\lim_{n \to \infty} \left| \frac{\frac{(4x-1)^{n+1}}{9^{n+1}(n+1)}}{\frac{(4x-1)^n}{9^n(n-1)}} \right| = \lim_{n \to \infty} \left| \frac{(4x-1)(n-1)}{9n} \right|
\]

\( = \left| \frac{4x-1}{9} \right| < 1 \)
\( |4x-1| < 9 \)
\( |x - \frac{1}{4}| < \frac{9}{4} \) \( \Rightarrow \) Center at \( x = \frac{1}{4} \), \( R = \frac{9}{4} \)

**For \( x = -2 \):**
\[
\sum_{n=2}^{\infty} \frac{(-1)^n(-9)^n}{9^n(n-1)} = \sum_{n=2}^{\infty} \frac{(-1)^n(-1)^n9^nn}{9^n(n-1)} = \sum_{n=2}^{\infty} \frac{1}{n-1} \geq \sum_{n=2}^{\infty} \frac{1}{n} \]
Diverges by Comparison with the harmonic series.

**For \( x = \frac{5}{2} \):**
\[
\sum_{n=2}^{\infty} \frac{(-1)^n(9)^n}{9^n(n-1)} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n-1} , \text{ which converges by AST.}
\]

\[ I = \left[ -2, \frac{5}{2} \right] \]

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<thead>
<tr>
<th>Question</th>
<th>Points Awarded</th>
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<tbody>
<tr>
<td>1-15</td>
<td></td>
<td>60</td>
</tr>
<tr>
<td>16</td>
<td></td>
<td>8</td>
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10