MATH 152, Spring 2021

COMMON EXAM II - VERSION **A**

LAST NAME(print): Solutions FIRST NAME(print): ____________

INSTRUCTOR: ____________

SECTION NUMBER: ____________

**DIRECTIONS:**

1. The use of a calculator, notes, or non-approved webpages is prohibited.
2. For multiple choice questions, the answer choices are in no particular order. You will need to select the corresponding answer choice in eCampus/Canvass.
3. For workout questions, work these problems on the answer template that was provided by your instructor. If your instructor did not provide the template, use your own paper. Don't forget to write your Name and UIN at the top of the page for every work out question. Ignore the yes/no choice at the end.
4. When you are done with the exam, you will use your phone to scan the solutions to the workout questions into a single pdf file and submit this pdf file to Gradescope. **Only submit the solutions to the workout questions. Do not include any solutions to the multiple choice.**
5. Show all your work neatly and concisely. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.
6. You are expected to follow THE AGGIE CODE OF HONOR **"An Aggie does not lie, cheat or steal, or tolerate those who do."**

## PART I: Multiple Choice. 4 points each.

1. Find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{3^{n+1}}$.

(a) $\frac{-2}{15}$

(b) $\frac{-1}{3}$

(c) $1$

(d) $\frac{2}{15}$

(e) $\frac{1}{3}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{3 \cdot 3^n} = \sum_{n=1}^{\infty} \frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)^{n-1}$$

$$= \frac{-\frac{2}{9}}{1+\frac{2}{3}}$$

$$= \frac{-2/9}{5/3} \quad \text{or} \quad \frac{-2}{9}\cdot\frac{3}{5} = \boxed{-\frac{2}{15}}$$

2. The sequence $a_n = 2\ln(n) - \ln(3n^2+5)$

(a) Converges to $\ln\left(\frac{1}{3}\right)$

(b) Diverges

(c) Converges to $\ln 3$

(d) Converges to $0$

(e) Converges to $2\ln\left(\frac{1}{3}\right)$

$$\lim_{n\to\infty}\left(2\ln(n) - \ln(3n^2+5)\right)$$

$$\lim_{n\to\infty}\left(\ln(n)^2 - \ln(3n^2+5)\right)$$

$$\lim_{n\to\infty} \ln\left(\frac{n^2}{3n^2+5}\right) = \boxed{\ln\left(\frac{1}{3}\right)}$$

3. $\int \frac{3-x}{x^2+3x-4}\,dx =$

(a) $\frac{-7}{5}\ln|x+4| + \frac{2}{5}\ln|x-1| + C$

(b) $\frac{7}{2}\ln|x+4| + \frac{2}{5}\ln|x-1| + C$

(c) $\frac{1}{5}\ln|x-4| - \frac{4}{5}\ln|x+1| + C$

(d) $\frac{-1}{5}\ln|x-4| + \frac{4}{5}\ln|x+1| + C$

(e) $\frac{2}{5}\ln|x+4| - \frac{7}{5}\ln|x-1| + C$

PFD: $\frac{3-x}{(x+4)(x-1)} = \frac{A}{x+4} + \frac{B}{x-1}$

$3-x = A(x-1) + B(x+4)$

$x=1$: $2 = B(5)$ $B = \frac{2}{5}$

$x=-4$: $7 = A(-5)$ $A = \frac{-7}{5}$

$$\int\left(\frac{-7/5}{x+4} + \frac{2/5}{x-1}\right)dx = \boxed{-\frac{7}{5}\ln|x+4| + \frac{2}{5}\ln|x-1| + C}$$

4. If we use the appropriate trigonometric substitution to evaluate $\int_1^{2/\sqrt{3}} \left(\frac{\sqrt{x^2-1}}{x}\right) dx$, which of the following is the correct result?

(a) $\int_0^{\pi/6} \tan^2\theta \, d\theta$

(b) $\int_0^{\pi/6} \frac{\tan\theta}{\sec\theta} \, d\theta$

(c) $\int_0^{\pi/3} \tan^2\theta \, d\theta$

(d) $\int_{\pi/2}^{\pi/6} \sin^2\theta d\theta$

(e) $\int_{\pi/2}^{\pi/3} \sin^2\theta d\theta$

$x = \sec\theta$ $\begin{cases} x = 2/\sqrt{3}, & \theta = \frac{\pi}{6} \\ x = 1, & \theta = 0 \end{cases}$

$dx = \sec\theta\tan\theta\, d\theta$

$$\int_0^{\frac{\pi}{6}} \frac{\sqrt{\sec^2\theta - 1}}{\cancel{\sec\theta}} \cancel{\sec\theta}\tan\theta\, d\theta$$

$$\boxed{\int_0^{\frac{\pi}{6}} \tan^2\theta\, d\theta}$$

5. Let $s = \sum_{n=1}^{\infty} \frac{3}{n^4}$. Using the Remainder Estimate for the Integral Test, find the smallest value of $n$ that will ensure that $R_n = s - s_n \leq \frac{1}{100}$.

(a) $n = 5$

(b) $n = 4$

(c) $n = 8$

(d) $n = 6$

(e) $n = 3$

$R_n \leq \int_n^{\infty} f(x)\, dx$ where $f(n) = a_n = \frac{3}{n^4}$

$R_n \leq \int_n^{\infty} \frac{3}{x^4}\, dx = \left.\frac{-1}{x^3}\right|_n^{\infty}$

$= \frac{1}{n^3}$. The smallest value that makes $\frac{1}{n^3} \leq \frac{1}{100}$ is $\boxed{n=5}$

6. Which of the following sequences are convergent? Assume $n \geq 1$.

(I) $a_n = \frac{(-1)^n n}{n+1}$ D

(II) $a_n = \frac{\ln(n^3)}{n}$ C

(III) $a_n = \frac{2n}{1+n}$ C

(a) (II) and (III) only.

(b) (III) only.

(c) (I) and (III) only.

(d) (II) only.

(e) All three are convergent.

$\lim_{n\to\infty} \frac{(-1)^n n}{n+1}$ diverges by oscillation

$\lim_{n\to\infty} \frac{\ln(n^3)}{n} = \lim_{n\to\infty} \frac{3\ln(n)}{n} \leq \lim_{n\to\infty} \frac{3}{n} = 0$

$\lim_{n\to\infty} \frac{2n}{1+n} = 2$ $\boxed{\text{only II \& III}}$

7. Evaluate $\int_0^1 \frac{x+1}{x+2}\, dx$.

(a) $1 - \ln(3) + \ln(2)$

(b) $1 + \ln(2)$

(c) $1 + \ln(3) + \ln(2)$

(d) $\ln(3) - \ln(2)$

(e) $1 - \ln(3) - \ln 2$

long division: $x+2 \overline{) x+1}$ quotient $1$; $-(x+2)$; remainder $-1$

$$\int_0^1 \left(1 - \frac{1}{x+2}\right) dx = \left(x - \ln|x+2|\right)\Big|_0^1 = \boxed{1 - \ln 3 + \ln 2}$$

8. Which of the following statements is true regarding the improper integral $\int_1^\infty \frac{dx}{x(\sqrt{x}+1)}$?

(a) Converges since $\int_1^\infty \frac{dx}{x(\sqrt{x}+1)} < \int_1^\infty \frac{dx}{x^{3/2}}$, which converges.

(b) Converges since $\int_1^\infty \frac{dx}{x(\sqrt{x}+1)} < \int_1^\infty \frac{dx}{x^2}$, which converges.

(c) Diverges since $\int_1^\infty \frac{dx}{x(\sqrt{x}+1)} < \int_1^\infty \frac{dx}{x}$, which diverges.

(d) Diverges since $\int_1^\infty \frac{dx}{x(\sqrt{x}+1)} < \int_1^\infty \frac{dx}{\sqrt{x}}$, which diverges.

(e) Converges since $\int_1^\infty \frac{dx}{x(\sqrt{x}+1)} > \int_1^\infty \frac{dx}{x^{3/2}}$, which converges.

$\int_1^\infty \frac{dx}{x(\sqrt{x}+1)} < \int_1^\infty \frac{dx}{x\sqrt{x}} = \int_1^\infty \frac{dx}{x^{3/2}}$, which converges since $p = 3/2 > 1$.

9. Which of the following series diverges by the **Test for Divergence**?

(a) $\sum_{n=2}^\infty \frac{n}{\sqrt{n^2+1}}$

(b) $\sum_{n=2}^\infty \frac{1}{n}$

(c) $\sum_{n=2}^\infty \sin\left(\frac{1}{n}\right)$

(d) $\sum_{n=2}^\infty \arctan\left(\frac{1}{n}\right)$

(e) $\sum_{n=2}^\infty \frac{\ln n}{n}$

For $\sum_{n=2}^\infty \frac{n}{\sqrt{n^2+1}}$, $\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = 1 \neq 0$, series diverges by T.D.

For all other series $\sum a_n$, $\lim_{n\to\infty} a_n = 0$, so T.D. fails

10. If the sequence of partial sums of the series $\sum_{n=1}^\infty a_n$ is $s_n = \frac{(2n+1)(n+1)}{n^2}$, find $a_2$.

(a) $\frac{-9}{4}$

(b) $\frac{-23}{36}$

(c) $\frac{15}{4}$

(d) $\frac{51}{4}$

(e) 2

$a_2 = S_2 - S_1$

$= \frac{15}{4} - 6$

$= -\frac{9}{4}$

11. Which of the following integrals are improper?

(I) $\int_1^e \frac{1}{\ln x}\,dx$ improper since $\ln(1)=0$

(II) $\int_{-2}^2 \frac{1}{x+1}\,dx$ improper at $x=-1$, since $x=-1$ is a vertical asymptote

(III) $\int_{-\infty}^0 \frac{1}{x^2+1}\,dx$ improper since $-\infty$ is a limit of integration

(a) All of them are improper

(b) II and III only

(c) III only

(d) I and III only

$\boxed{\text{all are improper}}$

12. $\int_{e^2}^{\infty} \frac{1}{x(\ln x)^3}\,dx =$

Let $u = \ln x$, $du = \frac{1}{x}dx$. Then $\int \frac{dx}{x(\ln x)^3} = \int \frac{du}{u^3} = \frac{-1}{2u^2} = \frac{-1}{2(\ln x)^2}$

(a) $\frac{1}{8}$

(b) 0

(c) $\frac{1}{4}$

(d) $\frac{-1}{4}$

(e) $\frac{-1}{8}$

$$\int_{e^2}^{\infty} \frac{dx}{x(\ln x)^3} = \left.\frac{-1}{2(\ln x)^2}\right|_{x=e^2}^{x=\infty}$$

$$= \frac{-1}{2(\infty)^2} + \frac{1}{2(\ln e^2)^2} = \frac{1}{2(4)} = \boxed{\frac{1}{8}}$$

13. The sequence $a_n = \dfrac{\ln(2+e^n)}{3n}$

(a) Converges to $\frac{1}{3}$

(b) Converges to 0.

(c) Converges to $\frac{2}{3}$

(d) Converges to 1.

(e) Diverges

$$\lim_{n\to\infty} \frac{\ln(2+e^n)}{3n} = \frac{\infty}{\infty} \overset{L}{=} \lim_{n\to\infty} \frac{\frac{e^n}{2+e^n}}{3}$$

$$= \lim_{n\to\infty} \frac{1}{3}\left(\frac{e^n}{2+e^n}\right) = \boxed{\frac{1}{3}}$$

14. Find the sum of the series $\sum_{n=1}^{\infty} a_n$ if the sequence of partial sums of the series $\sum_{n=1}^{\infty} a_n$ is $s_n = \dfrac{\pi n+1}{n} + 1$.

(a) $\sum_{n=1}^{\infty} a_n = \pi + 1$

(b) $\sum_{n=1}^{\infty} a_n = \pi(\pi+1)$

(c) $\sum_{n=1}^{\infty} a_n = 1$

(d) $\sum_{n=1}^{\infty} a_n = 0$

(e) $\sum_{n=1}^{\infty} a_n$ diverges

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$$

$$= \lim_{n\to\infty} \left(\frac{\pi n + 1}{n} + 1\right)$$

$$= \boxed{\pi + 1}$$

15. The recursive sequence defined below is both bounded and increasing. Which of the following statements is true?

$a_1 = 3$; $a_{n+1} = \sqrt{6a_n - 5}$

Bounded & increasing means it has a limit, $L$.

$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left(\sqrt{6a_n - 5}\right) \to L = \sqrt{6L-5}$,

So $L^2 = 6L - 5 \to L^2 - 6L + 5 = 0$

$\to (L-5)(L-1) = 0$

$L = 5$, since $a_1 = 3$ & sequence increases

(a) The sequence converges to 5.

(b) The sequence converges to 3.

(c) The sequence converges to 1.

(d) The sequence converges to 6.

(e) The sequence diverges.

**PART II: Free Response: Show all work and box your final answer!**

16. (10 pts) Consider $\sum_{n=1}^{\infty}\left(\frac{2}{2n-1}-\frac{2}{2n+1}\right)$.

a.) (6 pts) Find the general formula for $s_n$, the sequence of partial sums.

b.) (4 pts) Find the sum of the series.

a.) $S_n = \sum_{i=1}^{n}\left(\frac{2}{2i-1}-\frac{2}{2i+1}\right)$

$= \underbrace{2-\frac{2}{3}}_{a_1}+\underbrace{\frac{2}{3}-\frac{2}{5}}_{a_2}+\underbrace{\frac{2}{5}-\frac{2}{6}}_{a_3}+\dots+\frac{2}{2n-1}-\frac{2}{2n+1}$

$\boxed{S_n = 2-\frac{2}{2n+1}}$

b.) $\sum_{n=1}^{\infty}\left(\frac{2}{2n-1}-\frac{2}{2n+1}\right)=\lim_{n\to\infty}\left(2-\frac{2}{2n+1}\right)$

$=\boxed{2}$

Form $a^2 - x^2 \to x = a\sin\theta$

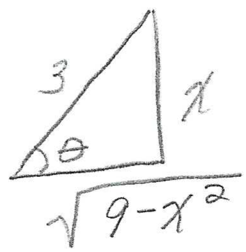
17. (12 pts) Find $\int \frac{x^2}{\sqrt{9-x^2}}\,dx$ showing all necessary work. Note: Your final answer must be in terms of $x$, as shown in class.

Let $x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

Reference triangle:

$\frac{x}{3} = \sin\theta$




and $\theta = \arcsin\left(\frac{x}{3}\right)$

$$\int \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}} \cdot 3\cos\theta\, d\theta$$

$$= 27\int \frac{\sin^2\theta\cos\theta}{\sqrt{9(1-\sin^2\theta)}}\, d\theta$$

$$= 27\int \frac{\sin^2\theta\,\cancel{\cos\theta}}{3\cancel{\cos\theta}}\, d\theta$$

$$= 9\int \sin^2\theta\, d\theta$$

$$= 9\int \frac{1}{2}(1-\cos 2\theta)\, d\theta$$

(note: $\sin(2\theta) = 2\sin\theta\cos\theta$)

$$= \frac{9}{2}\left(\theta - \frac{1}{2}\sin(2\theta)\right) + C$$

$$= \frac{9}{2}(\theta - \sin\theta\cos\theta) + C = \boxed{\frac{9}{2}\left(\arcsin\left(\frac{x}{3}\right) - \frac{x}{3}\frac{\sqrt{9-x^2}}{3}\right) + C}$$

18. (10 pts) Find $\int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)}\,dx$ showing all necessary work.

$$\frac{2x^3-4x-8}{x(x-1)(x^2+4)} = \frac{A}{x} + \frac{B}{x-1} + \frac{cx+d}{x^2+4}$$

$$2x^3-4x-8 = A(x-1)(x^2+4) + Bx(x^2+4) + (cx+d)x(x-1)$$

$$x=0: \quad -8 = A(-4) \quad \boxed{A=2}$$

$$x=1: \quad -10 = B(5) \quad \boxed{B=-2}$$

$$2x^3-4x-8 = 2(x-1)(x^2+4) - 2x(x^2+4) + (cx+d)(x^2-x)$$

$$= 2(x^3+4x-x^2-4) - 2x^3 - 8x + cx^3 - cx^2 + dx^2 - dx$$

$$= \cancel{2x^3} + \cancel{8x} - 2x^2 - 8 - \cancel{2x^3} - \cancel{8x} + cx^3 - cx^2 + dx^2 - dx$$

$$= cx^3 + (-2-c+d)x^2 - dx - 8$$

$$\boxed{c=2}$$

$$0 = -2-c+d$$

$$0 = -4+d$$

$$\boxed{d=4}$$

$$\int\left(\frac{2}{x} - \frac{2}{x-1} + \frac{2x+4}{x^2+4}\right)dx$$

$\leftarrow$ u-sub $\quad \int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$, where $a=2$

$$\int\left(\frac{2}{x} - \frac{2}{x-1} + \frac{2x}{x^2+4} + \frac{4}{x^2+4}\right)dx$$

$$= \boxed{2\ln|x| - 2\ln|x-1| + \ln(x^2+4) + 2\arctan\left(\frac{x}{2}\right) + C}$$

19. (8 pts) Use the Integral Test to determine whether $\sum_{n=1}^{\infty} ne^{-4n^2}$ converges or diverges. Support your answer.

Let $f(x) = xe^{-4x^2}$.

note: $f(x)$ is continuous, positive and decreasing for $x \geq 1$.

Consider $\int_1^{\infty} xe^{-4x^2}\,dx$

$u = -4x^2$

$du = -8x\,dx$

$\int xe^{-4x^2}\,dx = -\frac{1}{8}\int e^u\,du$

$= -\frac{1}{8}e^u$

$= -\frac{1}{8}e^{-4x^2}$

$$\int_1^{\infty} xe^{-4x^2}\,dx = \lim_{t\to\infty}\int_1^t xe^{-4x^2}\,dx$$

$$= \lim_{t\to\infty} -\frac{1}{8}e^{-4x^2}\Big|_1^t$$

$$= \lim_{t\to\infty} -\frac{1}{8}\left(e^{-4t^2} - e^{-4}\right)$$

$$= \frac{1}{8}(e^{-4})$$

$\boxed{\text{Since the improper integral converges so does the series by the integral test}}$