MATH 152, Spring 2021

COMMON EXAM III - VERSION A

LAST NAME(print): solutions FIRST NAME(print): ____________

INSTRUCTOR: ____________

SECTION NUMBER: ____________

DIRECTIONS:

1. The use of a calculator, notes, or non-approved webpages is prohibited.
2. For multiple choice questions, the answer choices are in no particular order. You will need to select the corresponding answer choice in eCampus/Canvass.
3. For workout questions, work these problems on the answer template that was provided by your instructor. If your instructor did not provide the template, use your own paper. Don't forget to write your Name and UIN at the top of the page for every work out question. Ignore the yes/no choice at the end.
4. When you are done with the exam, you will use your phone to scan the solutions to the workout questions into a single pdf file and submit this pdf file to Gradescope. **Only submit the solutions to the workout questions. Do not include any solutions to the multiple choice.**
5. Show all your work neatly and concisely. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.
6. You are expected to follow THE AGGIE CODE OF HONOR **"An Aggie does not lie, cheat or steal, or tolerate those who do."**

note: For Q12, since there are two correct answers, everyone got full credit

PART I: Multiple Choice. 4 points each.

1. If we use s_4 to approximate $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$, use the Alternating Series Estimation Theorem to estimate the absolute value of the error, $|R_4|$.

(a) $|R_4| \leq \frac{1}{625}$

(b) $|R_4| \leq \frac{1}{81}$

(c) $|R_4| \leq \frac{1}{375}$

(d) $|R_4| \leq \frac{1}{256}$

(e) $|R_4| \leq \frac{1}{192}$

$|R_n| < a_{n+1}$, where $a_n = \frac{1}{n^4}$

$|R_4| \leq a_5 = \frac{1}{5^4} = \boxed{\frac{1}{625}}$

2. $\sum_{n=1}^{\infty} \frac{4}{3^n + \sqrt{n}}$ is:

(a) Convergent by the Comparison Test with $\sum_{n=1}^{\infty} \frac{4}{3^n}$.

(b) Convergent by the Comparison Test with $\sum_{n=1}^{\infty} \frac{4}{\sqrt{n}}$.

(c) Divergent by the Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{4}{3^n}$.

(d) Divergent by the Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{4}{\sqrt{n}}$.

(e) Divergent by the Integral Test.

Comparison Test:

$\frac{4}{3^n+\sqrt{n}} < \frac{4}{3^n}$, and

$\sum_{n=1}^{\infty} \frac{4}{3^n}$ is a convergent geometric series, so $\sum_{n=1}^{\infty} \frac{4}{3^n+\sqrt{n}}$ also converges.

3. For which of the following series is the Ratio Test inconclusive?

I) $\sum_{n=1}^{\infty} \frac{(-3)^n}{n^2+3}$

II) $\sum_{n=2}^{\infty} \frac{n}{5n^4+3}$

III) $\sum_{n=1}^{\infty} \frac{4^n}{(n+1)(-4)^n}$

(a) II and III only.

(b) II only.

(c) III only.

(d) I and II only.

(e) I, II, and III.

I: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty > 1$ diverges by ratio test

II. $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, ratio test inconclusive

III $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, ratio test inconclusive.

$\boxed{\text{II \& III only}}$

4. The series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\sqrt[3]{n^3+1}}$

$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n\sqrt[3]{n^3+1}} \right| = \sum_{n=2}^{\infty} \frac{1}{n\sqrt[3]{n^3+1}} \leq \sum_{n=2}^{\infty} \frac{1}{n^2},$

which converges by p-series,

Thus $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\sqrt[3]{n^3+1}}$ converges absolutely

(a) Converges absolutely.

(b) Converges but not absolutely.

(c) Diverges by the test for divergence.

(d) Diverges by the alternating series test.

(e) Diverges by p series.

5. For what values of p does the series $\sum_{n=1}^{\infty} \frac{n^2}{1+n^p}$ converge?

If $p>3$, then $\sum_{n=1}^{\infty} \frac{n^2}{1+n^p}$ is comparable to a p-series where $p>1$, thus will converge

(a) only if $p > 3$.

(b) only if $p > 1$.

(c) only if $p > 4$.

(d) only if $p > 0$.

(e) only if $p > 2$.

6. The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$

comparison test fails since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1} < \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which diverges. By limit comparison test with $b_n = \frac{1}{\sqrt{n}}$, $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{\frac{1}{\sqrt{n}+1}}{\frac{1}{\sqrt{n}}} \right)$

$= \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+1}$, so both series diverge.

(a) Diverges by the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

(b) Converges since $\lim_{n\to\infty} \frac{1}{\sqrt{n}+1} = 0$.

(c) Diverges by the comparison test with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

(d) Diverges by test for divergence.

(e) Converges by the integral test.

7. The Ratio Test, when applied to the series $\sum_{n=1}^{\infty} \frac{(-10)^n (n+5)!}{(2n+1)!}$,

gives $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| =$

$$\lim_{n\to\infty} \left| \frac{(-10)^{n+1}(n+6)!}{(2n+3)!} \cdot \frac{(2n+1)!}{(-10)^n (n+5)!} \right|$$

(a) 0

(b) $\frac{1}{2}$

(c) 10

(d) ∞

(e) 5

$$\lim_{n\to\infty} \left| \frac{\cancel{(-10)^n}(-10)(n+6)\cancel{(n+5)!}}{(2n+3)(2n+2)\cancel{(2n+1)!}} \cdot \frac{\cancel{(2n+1)!}}{\cancel{(-10)^n}\cancel{(n+5)!}} \right| = 0$$

8. Which of the following statements is always true?

I) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ diverges by p-series. this series converges by alternating series test.

II) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=2}^{\infty} a_n$ converges. not necessarily true.

III) If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges absolutely. diefinition of absolute convergence.

(a) III only.

(b) II and III only.

(c) I and II only.

(d) I only.

(e) I and III only.

9. Find the third degree Taylor Polynomial, $T_3(x)$, for $f(x) = \sin(2x)$ at $a = \frac{\pi}{12}$.

(a) $T_3(x) = \frac{1}{2} + \sqrt{3}\left(x - \frac{\pi}{12}\right) - \left(x - \frac{\pi}{12}\right)^2 - \frac{2\sqrt{3}}{3}\left(x - \frac{\pi}{12}\right)^3$ $\quad f(x) = \sin(2x),\ f(\frac{\pi}{12}) = \frac{1}{2}$

(b) $T_3(x) = \frac{1}{2} + \sqrt{3}\left(x - \frac{\pi}{12}\right) - 2\left(x - \frac{\pi}{12}\right)^2 - 4\sqrt{3}\left(x - \frac{\pi}{12}\right)^3$ $\quad f'(x) = 2\cos(2x),\ f'(\frac{\pi}{12}) = \sqrt{3}$

(c) $T_3(x) = \frac{\sqrt{3}}{2} + \left(x - \frac{\pi}{12}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{12}\right)^2 - \frac{\sqrt{3}}{3}\left(x - \frac{\pi}{12}\right)^3$ $\quad f''(x) = -4\sin(2x),\ f''(\frac{\pi}{12}) = -2$

(d) $T_3(x) = \frac{1}{2} + \sqrt{3}\left(x - \frac{\pi}{12}\right) + \left(x - \frac{\pi}{12}\right)^2 - \frac{2\sqrt{3}}{3}\left(x - \frac{\pi}{12}\right)^3$ $\quad f'''(x) = -8\cos(2x),\ f'''(\frac{\pi}{12}) = -4\sqrt{3}$

(e) $T_3(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{12}\right) - \frac{1}{4}\left(x - \frac{\pi}{12}\right)^2 - \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{12}\right)^3$

$$T_3(x) = f(\tfrac{\pi}{12}) + f'(\tfrac{\pi}{12})(x - \tfrac{\pi}{12}) + \frac{f''(\frac{\pi}{12})}{2}(x - \tfrac{\pi}{12})^2 + \frac{f'''(\frac{\pi}{12})}{6}(x - \tfrac{\pi}{12})^3$$

10. Using the definition of Taylor Series, find $f^{(18)}(4)$, that is, the 18^{th} derivative of $f(x)$ at $x = 4$,

if $f(x) = \sum_{n=0}^{\infty} \frac{(-2)^n}{(2n+3)!}(x-4)^n$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(4)}{n!}(x-4)^n$$

(a) $\frac{(-2)^{18}(18)!}{(39)!}$

(b) 0

(c) $\frac{(-2)^{18}}{(18)!}$

(d) $\frac{(-2)^{18}}{(39)!}$

(e) $\frac{(-2)^{18}}{(21)!}$

Thus $\frac{f^{(n)}(4)}{n!} = \frac{(-2)^n}{(2n+3)!}$

$f^n(4) = \frac{n!(-2)^n}{(2n+3)!}$, so $f(4) = \frac{(18!)(-2)^{18}}{(39)!}$

11. $\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{5\pi}{6}\right)^{2n}}{(2n)!} =$

(a) $-\frac{\sqrt{3}}{2}$

(b) $-\frac{1}{2}$

(c) $\frac{\sqrt{3}}{2}$

(d) $\frac{1}{2}$

(e) $e^{5\pi/6}$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos\left(\frac{5\pi}{6}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{5\pi}{6}\right)^{2n}}{(2n)!} = -\frac{\sqrt{3}}{2}$$

$\downarrow$

$-\frac{\sqrt{3}}{2}$

12. Suppose it is known that $\sum_{n=0}^{\infty} c_n(x-1)^n$ converges when $x = 5$.

Which of the following is certain to be true?

$x=1$ is the center.

(a) $\sum_{n=0}^{\infty} c_n(-2)^n$ is convergent.

(b) $\sum_{n=0}^{\infty} c_n(6)^n$ is divergent.

(c) $\sum_{n=0}^{\infty} c_n(-3)^n$ is convergent.

(d) $\sum_{n=0}^{\infty} c_n 4^n$ is divergent.

(e) None of these are certain to be true.

don't know $\leftarrow$ (converges] $\rightarrow$ don't know

-3 $\quad$ 1 $\quad$ 5

4 units $\quad$ 4 units

• $\sum_{n=0}^{\infty} c_n(-2)^n$ is convergent since $x=-1$ is within convergent interval

• $\sum c_n(-3)^n$ is convergent since $x=-2$ is within interval

note: everyone got credit for #12 since there are two correct answers

13. $\int e^{-5x^2}\,dx =$

(a) $C + \sum_{n=0}^{\infty} \frac{(-1)^n 5^n x^{2n+1}}{(2n+1)n!}$

(b) $C + \sum_{n=0}^{\infty} \frac{(-1)^n 5^n x^{2n+2}}{(2n+2)n!}$

(c) $C + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 5^{n+1} x^{2n+2}}{(n+1)!}$

(d) $C + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 5^n x^{2n+2}}{(2n+2)n!}$

(e) $C + \sum_{n=0}^{\infty} \frac{(-1)^n 5^n x^{2n+6}}{(2n+6)n!}$

using $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$,

$\int e^{-5x^2}dx = \int \sum_{n=0}^{\infty} \frac{(-5x^2)^n}{n!}dx$

$= \int \sum_{n=0}^{\infty} \frac{(-1)^n 5^n x^{2n}}{n!}dx$

$= C + \sum_{n=0}^{\infty} \frac{(-1)^n 5^n x^{2n+1}}{n!(2n+1)}$

14. Write $f(x) = \frac{1}{2x+3}$ as a power series about zero.

(a) $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{3^{n+1}}$, where $|x| < \frac{3}{2}$.

(b) $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{3^{n+1}}$, where $|x| < \frac{2}{3}$.

(c) $f(x) = \sum_{n=0}^{\infty} \frac{2^n x^n}{3^{n+1}}$, where $|x| < \frac{3}{2}$.

(d) $f(x) = \sum_{n=0}^{\infty} \frac{2^n x^n}{3^{n+1}}$, where $|x| < \frac{2}{3}$.

(e) $f(x) = \sum_{n=0}^{\infty} \frac{2^n x^n}{3^{n-1}}$, where $|x| < \frac{3}{2}$.

$\frac{1}{3+2x} = \frac{1}{3(1+\frac{2x}{3})}$

$= \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{2x}{3}\right)^n,$

$\left|-\frac{2x}{3}\right| < 1$

$|x| < \frac{3}{2}$

$= \frac{1}{3}\sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{3^n}$

$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{3^{n+1}}, \quad |x| < \frac{3}{2}$

15. $\frac{\arctan(x^2)}{x} =$

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{2n+1}$

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{2n+1}$

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

(d) $\sum_{n=0}^{\infty} \frac{x^{4n+1}}{2n+1}$

(e) $\sum_{n=0}^{\infty} \frac{x^{4n}}{2n+1}$

using $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

$\frac{\arctan(x^2)}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{x(2n+1)}$

$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{x(2n+1)} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{2n+1}}$

PART II: Work Out: Support your answer!

16. For the power series $\sum_{n=1}^{\infty} \frac{(-2)^n (x+3)^n}{\sqrt{n+5}}$:

a.) (6 pts) Find the radius of convergence.

b.) (6 pts) Find the interval of convergence. You must test the endpoints for convergence.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}(x+3)^{n+1}}{\sqrt{n+6}} \quad \frac{\sqrt{n+5}}{(-2)^n(x+3)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-2)\cancel{(-2)^n}(x+3)\cancel{(x+3)^n}}{\sqrt{n+6}} \cdot \frac{\sqrt{n+5}}{\cancel{(-2)^n}\cancel{(x+3)^n}}\right|$$

$$= \left|-2(x+3)\right| \rightarrow |-2||x+3| < 1$$

$$\rightarrow 2|x+3| < 1$$

$$\rightarrow |x+3| < \tfrac{1}{2} \qquad \boxed{R = \tfrac{1}{2}}$$

$$\rightarrow -\tfrac{1}{2} < x+3 < \tfrac{1}{2}$$

$$\rightarrow -\tfrac{7}{2} < x < -\tfrac{5}{2}$$

endpoints: $x = -\frac{5}{2}$: $\sum_{n=1}^{\infty} \frac{(-2)^n(\frac{1}{2})^n}{\sqrt{n+5}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+5}}$

which converges by AST since $a_n = \frac{1}{\sqrt{n+5}}$ $\left\{ \begin{array}{l} \text{decreases,} \\ \lim_{n\to\infty} \frac{1}{\sqrt{n+5}} = 0 \end{array} \right.$

$x = -\frac{7}{2}$: $\sum_{n=1}^{\infty} \frac{(-2)^n(-\frac{1}{2})^n}{\sqrt{n+5}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+5}}$, which diverges by LCT, with $b_n = \frac{1}{\sqrt{n}}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{n+5}} \cdot \sqrt{n} = 1 > 0$

and $\sum \frac{1}{\sqrt{n}}$ diverges. Thus $\boxed{I = (-\frac{7}{2}, \frac{5}{2}]}$

17. (10 points) For $f(x) = \frac{1}{x^2}$, find the Taylor series at $a = 7$.

Note: You do not need to find the radius of convergence.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(7)}{n!}(x-7)^{n}$$

$$f(x) = \frac{1}{x^2}$$

$$f'(x) = -\frac{2}{x^3}$$

$$f''(x) = \frac{3 \cdot 2}{x^4}$$

$$f'''(x) = -\frac{4 \cdot 3 \cdot 2}{x^5}$$

$$\vdots$$

$$f^{n}(x) = \frac{(-1)^{n}(n+1)!}{x^{n+2}}, \text{ so } f^{n}(7) = \frac{(-1)^{n}(n+1)!}{7^{n+2}}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n}(n+1)!}{7^{n+2}\, n!}(x-7)^{n}, \quad \underline{\text{or}}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n}(n+1)}{7^{n+2}}(x-7)^{n}$$

18. (8 pts) Consider $\sum_{n=2}^{\infty} \frac{(-1)^n\sqrt{n}}{2n-1}$. Determine whether the series converges absolutely, converges conditionally (converges but not absolutely), or diverges.

Fully support your conclusion.

If we apply the alternating series test,

with $a_n = \frac{\sqrt{n}}{2n-1}$, this sequence is decreasing since, for

$$f(x) = \frac{\sqrt{x}}{2x-1}, \quad f'(x) = \frac{\frac{1}{2\sqrt{x}}(2x-1) - 2\sqrt{x}}{(2x-1)^2}$$

$$= \frac{2x-1-4x}{2\sqrt{x}(2x-1)^2}$$

$$= \frac{1-2x}{2\sqrt{x}(2x-1)^2} < 0 \text{ for } x \geq 2$$

and $\lim_{n\to\infty} \frac{\sqrt{n}}{2n-1} = 0$. However, the series does not converge absolutely since

$$\sum_{n=2}^{\infty} \frac{\sqrt{n}}{2n-1} > \sum_{n=2}^{\infty} \frac{\sqrt{n}}{2n} = \sum_{n=2}^{\infty} \frac{1}{2\sqrt{n}},$$ a divergent p-series.

Thus $\sum_{n=2}^{\infty} \frac{(-1)^n\sqrt{n}}{2n-1}$ converges conditionally

19. a.) (8 pts) Find a power series about zero for $f(x) = \ln(5 - 2x^2)$.

b.) (2 pts) What is the radius of convergence the series above?

method 1: $f(x) = \ln(5-2x^2) \rightarrow f'(x) = \dfrac{-4x}{5-2x^2}$

$$f'(x) = \frac{-4x}{5\left(1-\frac{2x^2}{5}\right)} = \frac{-4x}{5}\sum_{n=0}^{\infty}\left(\frac{2x^2}{5}\right)^n, \quad |x| < \sqrt{\frac{5}{2}}$$

$$= \frac{-4x}{5}\sum_{n=0}^{\infty}\frac{2^n x^{2n}}{5^n}$$

$$= -\frac{2^2 x}{5}\sum_{n=0}^{\infty}\frac{2^n x^{2n}}{5^n}$$

$$= -\sum_{n=0}^{\infty}\frac{2^{n+2}x^{2n+1}}{5^{n+1}}$$

now, $\ln(5-2x^2) = \int -\sum_{n=0}^{\infty}\frac{2^{n+2}x^{2n+1}}{5^{n+1}}$

let $x=0$, $c = \ln 5$

$$= C - \sum_{n=0}^{\infty}\frac{2^{n+2}x^{2n+2}}{5^{n+1}(2n+2)}$$

$$= \ln 5 - \sum_{n=0}^{\infty}\frac{\cancel{2}\cdot 2^{n+1}x^{2n+2}}{5^{n+1}\cancel{2}(n+1)}, \text{ so}$$

$$\boxed{\ln 5 - \sum_{n=0}^{\infty}\frac{2^{n+1}x^{2n+2}}{5^{n+1}(n+1)}, \quad R = \sqrt{5/2}}$$

19 method 2, using

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

$$\ln(5-2x^2) = \ln\left(5\left(1-\frac{2x^2}{5}\right)\right)$$

$$= \ln 5 + \ln\left(1-\frac{2x^2}{5}\right)$$

$$= \ln 5 + \sum_{n=0}^{\infty} \frac{(-1)^n \left(-\frac{2x^2}{5}\right)^{n+1}}{n+1}$$

$$= \ln 5 + \sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{n+1} 2^{n+1} x^{2n+2}}{5^{n+1}(n+1)}$$

$$= \ln 5 + \sum_{n=0}^{\infty} \frac{(-1)^{2n+1} 2^{n+1} x^{2n+2}}{5^{n+1}(n+1)}$$

note: $(-1)^{2n+1} = -1$

$$= \ln 5 - \sum_{n=0}^{\infty} \frac{2^{n+1} x^{2n+2}}{5^{n+1}(n+1)}$$