MATH 152, Spring 2022

EXAM II - VERSION A

LAST NAME(print): Key ______ FIRST NAME(print): ______

UIN: ______

INSTRUCTOR: ______

SECTION NUMBER: ______

DIRECTIONS:

1. No calculator, cell phones, or other electronic devices may be used, and they must all be put away out of sight.
2. TURN OFF cell phones and put them away. If a cell phone is seen during the exam, your exam will be collected and you will receive a zero.
3. In Part 1, mark the **correct** choice on your ScanTron using a No. 2 pencil. The scantrons will not be returned, therefore *for your own records, also record your choices on your exam!*
4. In Part 2, present your solutions in the space provided. *Show all your work* neatly and concisely and *clearly indicate your final answer.* You will be graded not merely on the final answer, but also on the quality and **correct**ness of the work leading up to it.
5. **Be sure to *fill in your name, UIN, section number and version letter of the exam on the ScanTron form.***

THE AGGIE CODE OF HONOR

"An Aggie does not lie, cheat or steal, or tolerate those who do."

Signature: ______

PART I: Multiple Choice. 4 points each.

1. Find the limit of the sequence $a_n = \arcsin\left(\frac{n+1}{1-2n}\right)$.

(a) $-\frac{\pi}{3}$

(b) $\frac{4\pi}{3}$

(c) $\frac{7\pi}{6}$

(d) $-\frac{\pi}{6}$ **correct**

(e) $\frac{\pi}{2}$

$$\lim_{n\to\infty} a_n = \arcsin\left(\lim_{n\to\infty} \frac{n+1}{1-2n}\right)$$

$$= \arcsin\left(-\frac{1}{2}\right)$$

$$= \frac{-\pi}{6}$$

2. Which of the following integrals are improper?

(I) $\int_0^1 \frac{1}{3x-1}\,dx$ (II) $\int_1^3 \ln(x-1)\,dx$ (III) $\int_{-\infty}^1 \frac{1}{x^4}\,dx$

(a) (III) only

(b) (I) and (III) only

(c) (II) and (III) only

(d) (I) and (II) only

(e) All of them are improper. **correct**

I. is improper since $\frac{1}{3x-1}$ has a vertical asymptote at $x = \frac{1}{3}$, which is on $[0,1]$.

II. is improper since $\ln(x-1)$ has a vertical asymptote at $x=1$, which is on $[1,3]$

III. is improper because $-\infty$ is a limit of integration

Thus all of them are improper.

3. The sequence $a_n = \frac{1}{3}\ln(4+2n) - \frac{1}{3}\ln(4n+1)$

(a) Converges to $\frac{1}{3}\ln\left(\frac{1}{2}\right)$ **correct**

(b) Diverges

(c) Converges to 0

(d) Converges to $\frac{1}{3}\ln(4)$

(e) Converges to $\frac{1}{3}\ln(2)$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{3}\left(\ln(4+2n) - \ln(4n+1)\right)$$

$$= \lim_{n\to\infty} \frac{1}{3}\ln\left(\frac{4+2n}{4n+1}\right)$$

$$= \frac{1}{3}\ln\left(\frac{1}{2}\right)$$

4. Which of the following series diverges by the Test for Divergence?

(a) $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

(b) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$

(d) $\sum_{n=1}^{\infty} e^{-n}$

(e) Test for Divergence fails for all of the above series. **correct**

a) $\lim_{n\to\infty} \ln\left(\frac{n}{n+1}\right) = \ln(1) = 0 \to$ Test fails

b) $\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = \sin(0) = 0 \to$ Test fails

c) $\lim_{n\to\infty} \frac{(-1)^n n}{n^2+1} = 0 \to$ Test fails

d) $\lim_{n\to\infty} e^{-n} = \lim_{n\to\infty} \frac{1}{e^n} = 0 \to$ Test fails

Divergence Test fails for all of them.

5. $\int \frac{x^3+x}{x-1}\, dx =$

Long division

$$\begin{array}{r} x^2 + x + 2 \\ x-1 \enclose{longdiv}{x^3 + 0x^2 + x} \\ -\ (x^3 - x^2) \\ \hline x^2 + x \\ -\ (x^2 - x) \\ \hline 2x \\ -\ (2x - 2) \\ \hline 2 \end{array}$$

(a) $\frac{x^3}{3}+\frac{x^2}{2} + 2x - 2\ln|x-1| + C$

(b) $\frac{x^3}{3}-\frac{x^2}{2} + 2x + 2\ln|x-1| + C$

(c) $\frac{x^3}{3}+\frac{x^2}{2} + 2x + 2\ln|x-1| + C$ **correct**

(d) $\frac{x^3}{3}+\frac{x^2}{2} - 2x + 2\ln|x-1| + C$

(e) $\frac{x^3}{3}-\frac{x^2}{2} + 2x - 2\ln|x-1| + C$

$$\int \left(x^2 + x + 2 + \frac{2}{x-1}\right) dx = \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x-1| + C$$

6. Evaluate $\int_1^{\infty} \frac{e^{2/x}}{x^2}\, dx.$ $= \lim_{t\to\infty} \int_1^t \frac{e^{2/x}}{x^2}\, dx$

Let $u = \frac{2}{x}$; $x=t, u=\frac{2}{t}$; $x=1, u=2$

$du = \frac{-2}{x^2}\, dx$

(a) $-\frac{1}{2}\left(1-e^2\right)$ **correct**

(b) $2\left(1-e^2\right)$

(c) $-2\left(1-e^2\right)$

(d) $\frac{1}{2}\left(1-e^2\right)$

(e) $\frac{1}{2}e^2$

$$= \lim_{t\to\infty} \int_2^{\frac{2}{t}} -\frac{1}{2} e^u\, du$$

$$= \lim_{t\to\infty} -\frac{1}{2}\left(e^{\frac{2}{t}} - e^2\right) \qquad e^0 = 1$$

$$= -\frac{1}{2}\left(1 - e^2\right)$$

7. Which of the following statements is/are true for the sequences shown below?

(I) $a_n = \cos\left(\frac{1}{n}\right)$ (II) $a_n = \frac{\cos(n)}{n}$

(a) (I) converges to 0, and (II) converges to zero.

(b) (I) converges to 1, and (II) diverges

(c) (I) diverges, and (II) converges to zero.

(d) (I) converges to 1, and (II) converges to zero. **correct**

(e) Both diverge

(I) $a_n = \cos\left(\frac{1}{n}\right)$

$\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos(0) = \boxed{1}$

(II) $a_n = \frac{\cos(n)}{n}$

cos(n) is bounded by ±1, hence is finite.

$\lim_{n\to\infty} \frac{\cos(n)}{n} = \frac{\text{finite \#}}{\infty} = \boxed{0}$

8. After an appropriate trigonometric substitution, $\int \frac{dx}{\sqrt{x^2+8x+41}}$ is equivalent to which of the following?

(a) $\frac{1}{5}\int \cos(\theta)\,d\theta$

(b) $\int \sec(\theta)\,d\theta$ **correct**

(c) $\int \sec^2(\theta)\,d\theta$

(d) $\int \tan(\theta)\,d\theta$

(e) $\frac{1}{5}\int \sin(\theta)\,d\theta$

complete square

$x^2 + 8x + 16 + 41 - 16 = (x+4)^2 + 25$

Let $x+4 = 5\tan\theta$

$dx = 5\sec^2\theta\,d\theta$

$\int \frac{dx}{\sqrt{(x+4)^2+25}}$

$\int \frac{5\sec^2\theta\,d\theta}{\sqrt{25\tan^2\theta+25}}$

$25(\tan^2\theta+1)$

$25\sec^2\theta$

$= \int \frac{5\sec^2\theta\,d\theta}{\sqrt{25\sec^2\theta}}$

$= \int \sec\theta\,d\theta$

9. Which of the following is a proper Partial Fraction Decomposition for $\frac{x+1}{(x^2-16)(x-3)^2(x^2+1)}$?

(a) $\frac{x+1}{(x^2-16)(x-3)^2(x^2+1)} = \frac{Ax+B}{x^2-16} + \frac{C}{x-3} + \frac{D}{(x-3)^2} + \frac{Ex+F}{x^2+1}$

(b) $\frac{x+1}{(x^2-16)(x-3)^2(x^2+1)} = \frac{A}{x-4} + \frac{B}{x+4} + \frac{C}{x-3} + \frac{D}{(x-3)^2} + \frac{Ex+F}{x^2+1}$ **correct**

(c) $\frac{x+1}{(x^2-16)(x-3)^2(x^2+1)} = \frac{Ax+B}{x^2-16} + \frac{Cx+D}{(x-3)^2} + \frac{Ex+F}{x^2+1}$

(d) $\frac{x+1}{(x^2-16)(x-3)^2(x^2+1)} = \frac{A}{x-4} + \frac{B}{x+4} + \frac{C}{x-3} + \frac{D}{(x-3)^2} + \frac{E}{x^2+1}$

(e) $\frac{x+1}{(x^2-16)(x-3)^2(x^2+1)} = \frac{A}{x-4} + \frac{B}{(x-4)^2} + \frac{C}{x-3} + \frac{D}{(x-3)^2} + \frac{Ex+F}{x^2+1}$

$\frac{x+1}{(x+4)(x-4)(x-3)^2(x^2+1)} = \frac{A}{x+4} + \frac{B}{x-4} + \frac{C}{x-3} + \frac{D}{(x-3)^2} + \frac{Ex+F}{x^2+1}$

10. Which of the following statements is true regarding the improper integral $\int_1^\infty \frac{\cos^2 x + 5}{x^4}\,dx$?

(a) It converges since $\int_1^\infty \frac{\cos^2 x + 5}{x^4}\,dx \le \int_1^\infty \frac{5}{x^4}\,dx$, which converges.

(b) It converges since $\int_1^\infty \frac{\cos^2 x + 5}{x^4}\,dx \le \int_1^\infty \frac{1}{x^4}\,dx$, which converges.

(c) It converges since $\int_1^\infty \frac{\cos^2 x + 5}{x^4}\,dx \le \int_1^\infty \frac{6}{x^4}\,dx$, which converges. **correct**

(d) It converges to zero.

(e) It diverges by oscillation.

ince $0 \le \cos^2 x \le 1$

$\cos^2 x + 5 \le 1 + 5 = 6$

$\int_1^\infty \frac{\cos^2 x + 5}{x^4}\,dx \le \int_1^\infty \frac{6}{x^4}\,dx$, which converges since $p = 4 > 1$.

11. After an appropriate trigonometric substitution, $\int_{2\sqrt{2}}^{4} \frac{\sqrt{x^2-4}}{x}\,dx$ is equivalent to

(a) $2\int_{\pi/4}^{\pi/3} \tan^2\theta\,d\theta$ **correct**

(b) $\int_{\pi/4}^{\pi/3} \sin(\theta)\,d\theta$

(c) $2\int_{\pi/4}^{\pi/6} \tan^2\theta\,d\theta$

(d) $\int_{\pi/4}^{\pi/6} \sin(\theta)\,d\theta$

(e) None of the above

$x = 2\sec\theta$

$x = 4 \rightarrow \frac{4}{2} = \sec\theta$, so $\theta = \frac{\pi}{3}$

$x = 2\sqrt{2} \rightarrow \sqrt{2} = \sec\theta$, so $\theta = \frac{\pi}{4}$

$dx = 2\sec\theta\tan\theta\,d\theta$

$4(\sec^2\theta - 1) = 4\tan^2\theta$

$\int_{\pi/4}^{\pi/3} \frac{\sqrt{4\sec^2\theta - 4}}{\cancel{2\sec\theta}}\,\cancel{2\sec\theta}\tan\theta\,\cancel{d\theta}$

$\int_{\pi/4}^{\pi/3} \sqrt{4\tan^2\theta}\,\tan\theta\,d\theta = 2\int_{\pi/4}^{\pi/3} \tan^2\theta\,d\theta$

12. Consider the recursive sequence $a_1 = 2$, $a_{n+1} = 5 - \frac{4}{a_n}$. Given the sequence is increasing and bounded, find the limit.

(a) 1

(b) $\frac{5}{2}$

(c) 2

(d) 4 **correct**

(e) 5

$a_1 = 2$

$a_{n+1} = 5 - \frac{4}{a_n}$

suppose $\lim_{n\to\infty} a_n = L$

then $\lim_{n\to\infty} a_{n+1} = L$

since L is the "end behavior" of a_n.

$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left(5 - \frac{4}{a_n}\right)$

$L = 5 - \frac{4}{L}$

$L^2 = 5L - 4$

$L^2 - 5L + 4 = 0$

$(L-4)(L-1) = 0$

$L = 4$ since a_n is

13. $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{4^n} =$ $\sum_{n=1}^{\infty} \frac{(-3)(-3)^n}{4^n} = \sum_{n=1}^{\infty} (-3)\left(-\frac{3}{4}\right)^n$

$= \sum_{n=1}^{\infty} (-3)\left(-\frac{3}{4}\right)\left(-\frac{3}{4}\right)^{n-1}$

(a) 9

(b) $\frac{2}{3}$

(c) $-\frac{12}{7}$

(d) $\frac{9}{7}$ **correct**

(e) -12

$r = -\frac{3}{4}$, which is less than one in absolute value, so series will converge

$\sum_{n=1}^{\infty} \frac{9}{4}\left(-\frac{3}{4}\right)^{n-1} = \frac{9/4}{1+\frac{3}{4}} = \frac{9}{7}$

14. Suppose $s_n = \frac{3n+4}{2n+2}$ is the sequence of partial sums for the series $\sum_{n=1}^{\infty} a_n$. Find $\sum_{n=1}^{\infty} a_n$ and a_4.

(a) $\sum_{n=1}^{\infty} a_n = \frac{3}{2}$ and $a_4 = -\frac{1}{60}$

(b) $\sum_{n=1}^{\infty} a_n = \frac{3}{2}$ and $a_4 = -\frac{129}{40}$

(c) $\sum_{n=1}^{\infty} a_n = 2$ and $a_4 = -\frac{1}{40}$

(d) $\sum_{n=1}^{\infty} a_n = \frac{3}{2}$ and $a_4 = -\frac{1}{40}$ **correct**

[redacted]

(e) $\sum_{n=1}^{\infty} a_n = 2$ and $a_4 = -\frac{1}{60}$,

$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{3n+4}{2n+2} = \frac{3}{2}$

$a_4 = S_4 - S_3 = \frac{16}{10} - \frac{13}{8} = -\frac{1}{40}$

Bubble 'a' for true and 'b' for false

15. If $\lim_{n\to\infty} s_n = 1$, then $\sum_{n=1}^{\infty} a_n$ diverges. **False** If $\lim_{n\to\infty} S_n = 1$, then $\sum_{n=1}^{\infty} a_n = 1$

16. If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges. **False** If $\lim_{n\to\infty} a_n = 0$, Test for divergence fails

17. The geometric series $\sum_{n=2}^{\infty} ar^{n-1}$ converges if $|r| < 1$. **True**

18. Given $f(x)$ and $g(x)$ are continuous, positive functions on the interval $[a, \infty)$ and $f(x) \geq g(x)$ on the interval $[a, \infty)$. If $\int_a^{\infty} g(x)\,dx$ diverges, so does $\int_a^{\infty} f(x)\,dx$. **True**

PART II: Free Response: Show all work and box your final answer!

19. (10 pts) Find $\int \frac{x+2}{x^2(x^2+1)}\,dx$

$$\frac{x+2}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

$$x+2 = Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2$$

$$x=0: \boxed{2=B}$$

$$x+2 = Ax^3 + Ax + 2x^2 + 2 + Cx^3 + Dx^2$$

$$x+2 = (A+C)x^3 + (2+D)x^2 + Ax + 2$$

$$0 = A+C$$

$$0 = 2+D \rightarrow \boxed{D=-2}$$

$$\boxed{1=A} \qquad \boxed{C=-1}$$

$$\int\left(\frac{1}{x} + \frac{2}{x^2} + \frac{-x-2}{x^2+1}\right)dx$$

$$\int\left(\frac{1}{x} + \frac{2}{x^2} + \frac{-x}{x^2+1} - \frac{2}{x^2+1}\right)dx$$

$$\ln|x| - \frac{2}{x} - \frac{1}{2}\ln(x^2+1) - 2\arctan x + C$$

20. (10 pts) Find $\int \frac{x^2}{\sqrt{4-x^2}}\,dx$

$x = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$

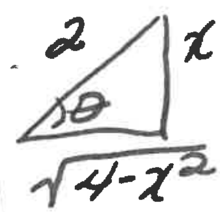
$$\int \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}}\,2\cos\theta\,d\theta$$

$$\int \frac{4\sin^2\theta}{\sqrt{4\cos^2\theta}}\,2\cos\theta\,d\theta$$

$$\int \frac{4\sin^2\theta}{2\cos\theta}\,2\cos\theta\,d\theta$$

$x = 2\sin\theta$

$\frac{x}{2} = \sin\theta$



$$\int 4\sin^2\theta\,d\theta$$

$$\int 4\cdot\frac{1}{2}(1-\cos 2\theta)\,d\theta$$

$2\sin\theta\cos\theta$

$$2\left(\theta - \frac{1}{2}\sin 2\theta\right) + C$$

$$\boxed{2(\theta - \sin\theta\cos\theta) + C}$$

$$2\left(\arcsin\frac{x}{2} - \frac{x}{2}\,\frac{\sqrt{4-x^2}}{2}\right) + C$$

21. Consider the series $\sum_{n=1}^{\infty} \frac{1}{(n+1)[\ln(n+1)]^2}$

(a) (6 pts) Prove the series converges.

integral test $f(x) = \frac{1}{(x+1)[\ln(x+1)]^2}$

which is a continuous, positive, decreasing function (C.P.D for short)

$$\int_1^{\infty} \frac{dx}{(x+1)[\ln(x+1)]^2} = \lim_{t\to\infty} \int_1^t \frac{dx}{(x+1)[\ln(x+1)]^2}$$

$u = \ln(x+1)$

$du = \frac{dx}{x+1}$

$$= \lim_{t\to\infty} \int_{\ln 2}^{\ln(t+1)} \frac{du}{u^2}$$

integral converges so does series

$$= \lim_{t\to\infty} -\frac{1}{u} \Big|_{\ln 2}^{\ln(t+1)} = \lim_{t\to\infty} \left[-\frac{1}{\ln(t+1)} + \frac{1}{\ln 2}\right]$$

(b) (4 pts) Using the Remainder Estimate for the Integral Test, find an upper bound on the remainder, R_8, if we used s_8, the 8^{th} partial sum, to approximate the sum of the series.

$$R_8 \leq \int_8^{\infty} \frac{dx}{(x+1)[\ln(x+1)]^2}$$

import integral from part a

$$R_8 \leq \lim_{t\to\infty} \int_8^t \frac{dx}{(x+1)[\ln(x+1)]^2}$$

$$\boxed{R_8 \leq \frac{1}{\ln 9}}$$

$$R_8 \leq -\frac{1}{\ln(x+1)} \Big|_8^{\infty} = \frac{1}{\ln 9}$$

22. Consider the series $\sum_{n=1}^{\infty} \frac{4}{n(n+2)}$

(a) (6 pts) Find a formula for s_n, the n^{th} partial sum.

$$\frac{4}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$$

$$4 = A(n+2) + Bn$$

$$n=0: \quad 4 = A(2) \quad A = 2$$

$$n=-2: \quad 4 = B(-2) \quad B = -2$$

$$\sum_{n=1}^{\infty}\left(\frac{2}{n} - \frac{2}{n+2}\right)$$ which telescopes

$$S_n = 2 - \cancel{\frac{2}{3}} + \frac{2}{2} - \cancel{\frac{2}{4}} + \cancel{\frac{2}{3}} - \cancel{\frac{2}{5}} + \cdots + \cancel{\frac{2}{n-1}} - \frac{2}{n+1} - \cancel{\frac{2}{n}} - \frac{2}{n+2}$$

$$\boxed{S_n = 2 + 1 - \frac{2}{n+1} - \frac{2}{n+2}}$$

(b) (4 pts) Find the sum of the series.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(3 - \frac{2}{n+1} - \frac{2}{n+2}\right)$$

$$= 3$$

DO NOT WRITE IN THIS TABLE.

Question	Points Awarded	Points
1-18		60
19		10
20		10
21		10
22		10
		100