Spring 1997 Math 152 Exam 2 Solutions Mon, 24/Mar/97

A-1b, B-1c [§8.5: Average Value of a Function] If the average value were $3/2$, the "area" of the part of the "hill" above the $y = 3/2$ cutoff would be equal to the sum of the areas of the "valleys" to the left and the right of the hill below the cutoff. This is clearly not the case from the picture; the area above the cutoff is less that that below the cutoff. Hence the average value of the function is less than $3/2$.

A-2e, B-4a [§8.6: Differential Equations] This is a *separable* differential equation. You know the drill… $2y\,dy = e^x\,dx \implies y^2 = e^x + C \implies y = +\sqrt{e^x + C}$, since $y > 0$.

A-3b, B-3c [§8.6: Differential Equations] Along the line $y = x$, the slope is $dy/dx = x - y = 0$. This narrows it down to two of the choices. At the point $(-1, 1)$, the slope is $dy/dx = x - y = -2 < 0$. This narrows it down to the correct choice.

A-4d, B-2e [§8.7: First-Order Linear Equations] First, divide by x to put the differential equation (DE) into standard linear form, $y' - \frac{3}{x}y = x$. From the initial condition (IC), $y(1) = 1$, we may presuppose that $x > 0$. Now multiply the DE by the integrating factor μ and solve. Here $\mu = \exp\left(\int -\frac{3}{x}\,dx\right) = e^{-3\ln|x|} = e^{\ln(|x|^{-3})} = |x|^{-3} = x^{-3}$, since $x > 0$.

$$\begin{aligned}
y' - \tfrac{3}{x}y &= x \\
x^{-3}y' - 3x^{-4}y &= x^{-2} \\
(x^{-3}y)' &= x^{-2} \\
x^{-3}y &= -x^{-1} + C \\
y &= Cx^3 - x^2. \quad \text{(Now use the IC.)} \\
1 = y(1) &= C - 1 \implies C = 2 \\
y &= 2x^3 - x^2
\end{aligned}$$

A-5a, B-6b [§8.8: Arc Length] Given $y = e^x \sin x$, we have that

$$\begin{aligned}
\text{arc length } = \int ds &= \int_a^b \sqrt{1 + (dy/dx)^2}\,dx \\
&= \int_0^{\pi/2} \sqrt{1 + (e^x\cos x + e^x \sin x)^2}\,dx \\
&= \int_0^{\pi/2} \sqrt{1 + (e^x(\cos x + \sin x))^2}\,dx \\
&= \int_0^{\pi/2} \sqrt{1 + e^{2x}(\cos^2 x + 2\cos x\sin x + \sin^2 x)}\,dx \\
&= \int_0^{\pi/2} \sqrt{1 + e^{2x} + 2e^{2x}\cos x \sin x}\,dx
\end{aligned}$$

A-6d, B-5e [§8.9: Area of a Surface of Revolution] Given $x = g(y) = (y-1)^2$, first compute the arc length differential, $ds = \sqrt{1 + (g'(y))^2}\,dy = \sqrt{1 + (2(y-1))^2}\,dy = \sqrt{1 + 4(y-1)^2}\,dy$. Then construct the surface area integral.

$$\begin{aligned}
\text{surface area } = \int 2\pi r\,ds &= \int 2\pi x\,ds \\
&= \int_1^2 2\pi(y-1)^2\sqrt{1 + 4(y-1)^2}\,dy
\end{aligned}$$

A-7c, B-9d [§9.1: Three-Dimensional Coordinate Systems] In $\mathbf{R}^3$, the equation $x^2 + z^2 = 4$, *with y unrestricted,* represents a cylinder of radius 2 whose axis of symmetry is the y-axis.

A-8e, B-8a [§9.2: Vectors] From the diagram, $\mathbf{B} + \mathbf{C} = \mathbf{A}$, whence $\mathbf{C} = \mathbf{A} - \mathbf{B}$. "Draw the picture, then do the algebra!"

A-9a, B-7b [§9.1: Three-Dimensional Coordinate Systems] First, the plane must contain the points $(0,2,0)$ and $(0,0,2)$. This eliminates choices $x+z=2, x+y=2$, and $x-y+z=2$. Next, the plane $x+y+z=2$ intersects the x-axis at $(2,0,0)$ *only*. Hence it is NOT parallel to the x-axis. Accordingly, the answer is $y+z=2$, wherein *x is unrestricted*.

A-10c, B-10d [§8.4: Work] First use Hooke's Law to determine the spring constant k: $F(x)=kx \Longrightarrow 20=k\cdot(4-2) \Longrightarrow k=10$. Now set up the work integral.

$$\text{work } W = \int_a^b F(x)\,dx = \int_{2-2}^{4-2} kx\,dx = \int_0^2 10x\,dx$$

A-11, B-11 [§8.7: First-Order Linear Equations] First multiply the differential equation by the integrating factor μ, then solve. Here $\mu = \exp\left(\int 1\,dt\right) = e^t$.

$$\begin{aligned}
y' + y &= t \\
e^t y' + e^t y &= te^t \\
(e^t y)' &= te^t \\
\text{Integrate RHS by parts.} \quad & u=t,\ \ du=dt,\ \ dv=e^t\,dt,\ \ v=e^t \\
e^t y &= te^t - e^t + C \\
y &= t-1+Ce^{-t} \quad \text{(Now use the IC.)} \\
1/2 = y(0) &= -1+C \Longrightarrow\ C=3/2 \\
y &= t-1+\tfrac{3}{2}e^{-t}
\end{aligned}$$

A-12, B-13 [§8.7: First-Order Linear Equations] Basically, *net rate* $=$ (*rate in*) $-$ (*rate out*), whence the net rate (in kg/min) is given by $dy/dt = (\frac{3}{10}\text{ kg/L}\cdot 10\text{ L/min}) - (\frac{y}{100}\text{ kg/L}\cdot 10\text{ L/min}) = 3-\frac{1}{10}y$, or $y'+\frac{1}{10}y=3$, a first-order differential equation in standard linear form. Multiply by the integrating factor, $\mu=\exp\left(\int \frac{1}{10}\,dt\right) = e^{\frac{1}{10}t}$, and then solve.

$$\begin{aligned}
e^{\frac{1}{10}t}y' + \tfrac{1}{10}e^{\frac{1}{10}t}y &= 3e^{\frac{1}{10}t} \\
(e^{\frac{1}{10}t}y)' &= 3e^{\frac{1}{10}t} \\
e^{\frac{1}{10}t}y &= 30e^{\frac{1}{10}t}+C \\
y &= 30+Ce^{-\frac{1}{10}t} \\
20 = y(0) &= 30+C \ \Longrightarrow\ C=-10 \\
y &= 30-10e^{-\frac{1}{10}t}
\end{aligned}$$

Alternatively, the differential equation $dy/dt = 3-\frac{1}{10}y$ is also separable. You may thus solve it using the techniques of §8.6. ("All roads lead to Rome.")

$$\begin{aligned}
\frac{dy}{3-\frac{1}{10}y} &= dt \\
-10\ln|3-\tfrac{1}{10}y| &= t+C_1 \\
\ln|3-\tfrac{1}{10}y| &= -\tfrac{1}{10}t+C_2 \\
|3-\tfrac{1}{10}y| &= e^{-\frac{1}{10}t+C_2} = C_3e^{-\frac{1}{10}t} \\
3-\tfrac{1}{10}y &= C_4e^{-\frac{1}{10}t} \\
3+C_5e^{-\frac{1}{10}t} &= \tfrac{1}{10}y \\
y &= 30+Ce^{-\frac{1}{10}t},
\end{aligned}$$

from which the rest is the same.

A-13, B-12 [§8.4: Work] Choose a conventional xy-coordinate system with positive x-values to the right and positive y-values upward. The bottom of the trough is at the origin. Express the width w of a layer of water in terms of y: $3/6 = w/y \Longrightarrow w = \frac{1}{2}y$. Build the differentials of volume, force, and work. Then set up and compute the work integral. Here $\delta = 62.5$ lb/ft^3.

$$\begin{aligned}
\text{volume differential, } dV &= lwh = 10\left(\tfrac{1}{2}y\right)\,dy = 5y\,dy \\
\text{force differential, } dF &= \delta\,dV = 5\delta y\,dy \\
\text{work differential, } dW &= dF\,z = (5\delta y\,dy)\cdot(6-y) = 5\delta y(6-y)\,dy \\
\text{work, } W &= \int_0^6 5\delta(6y - y^2)\,dy \\
&= 5\delta\left(3y^2 - \tfrac{1}{3}y^3\right)\Big|_0^6 = 5\delta(108 - 72) \\
&= 180\delta = 11,250 \text{ ft-lb}
\end{aligned}$$

A-14, B-16 [§8.10: Moments and Centers of Mass] The masses m_i are located at the points P_i. First find the total mass m of the system, then the moments M_x and M_y. Finally, compute the center of mass.

$$\text{total mass, } m = \sum_{i=1}^{3} m_i = 2+5+1 = 8$$

$$\text{moment w.r.t. } y\text{-axis, } M_y = \sum_{i=1}^{3} m_i x_i = (2)(1)+(5)(3)+(1)(4) = 2+15+4 = 21$$

$$\text{moment w.r.t. } x\text{-axis, } M_x = \sum_{i=1}^{3} m_i y_i = (2)(2)+(5)(1)+(1)(3) = 4+5+3 = 12$$

$$\text{center of mass, } CM = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \left(\frac{21}{8}, \frac{12}{8}\right) = \left(\frac{21}{8}, \frac{3}{2}\right)$$

A-15, B-14 [§8.11: Hydrostatic Pressure and Force] Choose a conventional xy-coordinate system with positive x-values to the right and positive y-values upward. The center of the semicircle is at the origin. Express the width w of a layer of water in terms of y: $w = 2x = 2\sqrt{1-y^2}$. Sequentially construct the area differential, pressure, and force differential. Then set up (and *optionally* compute) the force integral. Here $\delta = 62.5$ lb/ft^3 and z (below) is the depth of the water layer beneath the surface of the water.

$$\begin{aligned}
\text{area differential, } dA &= wh = 2\sqrt{1-y^2}\,dy \\
\text{pressure, } P &= \delta z = \delta(0-y) = -\delta y \\
\text{force differential, } dF &= P\,dA = -2\delta y\sqrt{1-y^2}\,dy \\
\text{hydrostatic force, } F &= \int_{-1}^{0} -2\delta y\sqrt{1-y^2}\,dy \\
&= \frac{2}{3}\delta(1-y^2)^{3/2}\Big|_{-1}^{0} = \frac{2}{3}\delta = \frac{125}{3} \text{ ft-lb} \approx 41.67 \text{ ft-lb}
\end{aligned}$$

A-16, B-15 [§8.10: Moments and Centers of Mass] The region is bounded above by $y = f(x) = \sqrt{1-x^2},\ 0 \le x \le 1$, below by the x-axis, $y = g(x) = 0,\ 0 \le x \le 1$, and on the left by the y-axis. Now plug and chug, employing brute force, calculator firepower, and/or ingenuity to get the job done!

Assuming uniform density ($\rho =$ constant, as always with the regions in §8.10), first find the total mass m, then the moments M_x and M_y. Finally, compute the the center of mass.

$$\begin{aligned}
\text{total mass, } m = \rho \int_a^b f(x) - g(x)\,dx &= \rho \int_0^1 \sqrt{1-x^2}\,dx = \rho A = \tfrac{1}{4}\pi\rho \\
\text{moment w.r.t. } y\text{-axis, } M_y = \rho \int_a^b x(f(x) - g(x))\,dx &= \rho \int_0^1 x\sqrt{1-x^2}\,dx \\
&= -\tfrac{1}{2}\rho \cdot \tfrac{2}{3}(1-x^2)^{3/2}\Big|_0^1 = 0 + \tfrac{1}{3}\rho = \tfrac{1}{3}\rho \\
\text{moment w.r.t. } x\text{-axis, } M_x = \rho \int_a^b \tfrac{1}{2}(f^2(x) - g^2(x))\,dx &= \rho \int_0^1 \tfrac{1}{2}(1-x^2)\,dx = \tfrac{1}{2}\rho(x - \tfrac{1}{3}x^3)\Big|_0^1 = \tfrac{1}{3}\rho \\
\text{center of mass, } CM = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) &= \left(\frac{\rho/3}{\pi\rho/4}, \frac{\rho/3}{\pi\rho/4}\right) = \left(\frac{4}{3\pi}, \frac{4}{3\pi}\right) \approx (0.42, 0.42)
\end{aligned}$$

NOTES: Techniques/insights to speed up your computations:

(a) Use the numerical or symbolic integration routines on your calculator. *"Firepower rules!"—Beavis*

(b) Recognize that $\int_0^1 \sqrt{1-x^2}\,dx$, the area of the quarter circular region, may be computed using geometry:

$$A = \frac{1}{4}\pi \cdot 1^2 = \frac{1}{4}\pi. \quad \text{(Also see the final note below.)}$$

(c) Notice that the region is symmetric about the line $y = x$, the $45°$ line. From this and the fact that the density of the region is uniform, we conclude that the center of mass lies on this line; i.e., $\overline{x} = \overline{y}$. Accordingly, we only need to compute ONE of the center of mass coordinates. Clearly $\overline{y}$ is easier since it involves M_x, in this case the integral of a polynomial.

(d) When all else fails, use a hand technique if you must. For example, if you didn't compute the area via geometry, use the trig substitution $x = \sin\theta,\ \ dx = \cos\theta\,d\theta$, as follows:

$$\int_0^1 \sqrt{1-x^2}\,dx = \int_0^{\pi/2} \cos^2\theta\,d\theta = \tfrac{1}{2}\int_0^{\pi/2} 1 + \cos 2\theta\,d\theta = \tfrac{1}{2}(\theta + \tfrac{1}{2}\sin 2\theta)\Big|_0^{\pi/2} = \tfrac{1}{4}\pi$$