Spring 1997 Math 152 Exam 3 Solutions Mon, 21/Apr/97

A-1b, B-2e [§§9.3–9.4 : The Dot and Cross Products] We cannot cross a scalar (dot product) with a vector.

A-2d, B-1b [§9.3: The Dot Product] With $\mathbf{b} = \langle 1, -7, 2\rangle$ and $\mathbf{a} = \langle 3, -2, -1\rangle$, the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \bullet \mathbf{b}}{||\mathbf{a}||} = \frac{3+14-2}{\sqrt{9+4+1}} = \frac{15}{\sqrt{14}} \approx 4.01$$

A-3c, B-3a [§9.6: Quadric Surfaces] The graph is that of a hyperboloid of two sheets, having standard form $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, or a permutation thereof. (That is, with the two minus signs possibly in front of other terms.) The key is that there be *two* minus signs with the constant side of the equation set to 1. Accordingly, this eliminates two of the choices. Of the remaining choices, only one contains the point $(0, 0, 1)$, which the picture shows is on the surface.

A-4a, B-5d [§9.7: Vector Functions and Space Curves]

The parameter value which corresponds to $P(4, 0, 2)$ is $t = 2$. Now plug and chug.

$$\begin{aligned} \text{tangent line: } \mathbf{L}(u) &= \mathbf{r}(2) + u\mathbf{r}'(2) \\ &= \langle 4, 0, 2\rangle + u\,\langle 2t, 3t^2 - 4, -4t^{-2}\rangle\big|_{t=2} \\ &= \langle 4, 0, 2\rangle + u\,\langle 4, 8, -1\rangle \\ &= \langle 4u + 4, 8u, 2 - u\rangle \qquad \text{or} \qquad \langle 4t + 4, 8t, 2 - t\rangle \end{aligned}$$

A-5d, B-4b [§9.9: Motion in Space: Velocity and Acceleration] Speed is the magnitude of velocity: $v = ||\mathbf{v}||$. *Constant* speed thus implies $v' = 0$, whence

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N} = v'\mathbf{T} + \kappa v^2\mathbf{N} = \kappa v^2\mathbf{N}$$

(since $v' = 0$). That is, the acceleration vector $\mathbf{a}$ points in the direction of the unit normal vector $\mathbf{N}$.

A-6c, B-7a [§10.1: Functions of Several Variables] Set $f(x, y) = c$, a nonzero constant; rearrange; then complete the square.

$$\begin{aligned} \frac{-2y}{x^2 + y^2 + 1} &= c \\ -\frac{2}{c}y &= x^2 + y^2 + 1 \\ \frac{1}{c^2} - 1 &= x^2 + y^2 + \frac{2}{c}y + \frac{1}{c^2} \\ x^2 + \left(y + \frac{1}{c}\right)^2 &= \frac{1}{c^2} - 1 = k = a^2 > 0 \quad \boxed{\text{circles}} \end{aligned}$$

(Positive sum of squares $\implies$ nondegenerate curves.)

A-7e, B-6c [§10.3: Partial Derivatives] Applying the Product Rule, we have

$$\frac{\partial}{\partial y}\left(y\sin(xy^2)\right) = y\cos(xy^2)\cdot 2xy + \sin(xy^2)\cdot 1 = \sin(xy^2) + 2xy^2\cos(xy^2)$$

A-8e, B-9c [§10.4: Tangent Planes and Differentials] With $z = f(x, y) = x^2 - y^2$ and $(x_0, y_0) = (2, 1)$, we have

$$\begin{aligned} \text{tangent plane:} \quad z &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ &= 3 + (2x)\big|_{(x,y)=(2,1)}(x - 2) + (-2y)\big|_{(x,y)=(2,1)}(y - 1) \\ &= 3 + 4(x - 2) - 2(y - 1) \qquad \text{i.e.,} \quad z = 4x - 2y - 3 \end{aligned}$$

A-9b, B-10d [§9.5: Equations of Lines and Planes] Equate the x-, y-, and z-slots and simultaneously solve.

$$\{t = 2s - 4, 6t = s - 2, -t = 8 - 4s\} \implies 6(2s - 4) = s - 2 \implies 11s = 22 \implies s = 2, t = 2s - 4 = 0$$

These values of s and t also satisfy the *third* equation. Therefore, the lines intersect at $(0, 0, 0)$. Assume the lines are perpendicular. Then their direction vectors are orthogonal. Hence their dot product is zero. But $\mathbf{v}_1 \bullet \mathbf{v}_2 = \langle 1, 6, -1\rangle \bullet \langle 2, 1, -4\rangle = 2 + 6 + 4 = 12$, a contradiction. Accordingly, the lines intersect, but are *not* perpendicular.

A-10a, B-8e [§10.3: Partial Derivatives] We numerically approximate the partial derivatives of f at the point $(2,2)$. In this regard, f is especially well-behaved with respect to x at $(2,2)$. The forward, backward, and central difference quotients all yield the same approximation. ("All roads lead to Rome…")

$$f_x(2,2) \approx \frac{-1-1}{3-2} = \frac{3-1}{1-2} = \frac{-1-3}{3-1} = -2$$

Only one of the five choices matches this approximation for $f_x(2,2)$, so we're done! But read on…

For the curious, a numerical approximation to $f_y(2,2)$ may be obtained by using the central difference quotient.

$$f_y(2,2) \approx \frac{-1-(-1)}{3-1} = 0$$

Of course, the textbook never spoke of central difference quotients! However, averaging the forward and backward difference quotients also yields zero.

$$\left(\frac{-1-1}{3-2} + \frac{-1-1}{1-2}\right) \Big/ 2 = (-2+2)/2 = 0$$

Alternatively, reason geometrically: as y increases along the line $x = 2$, the function values $f(2,y)$ increase, *level off*, then decrease. That is, they peak at $(2,2)$, signifying a local max of the curve $g(y) = f(2,y)$, at which the slope is zero.

A-11, B-13 [§9.3: The Dot Product] Place the cube in the first octant with one end of the diagonal at the origin. Then simply compute the angle between the vectors $\mathbf{v} = \langle 1,1,0\rangle$ (diagonal along the bottom face) and $\mathbf{w} = \langle 1,1,1\rangle$ (full diagonal).

$$\theta = \cos^{-1}\left(\frac{\mathbf{v}\bullet\mathbf{w}}{||\mathbf{v}||\,||\mathbf{w}||}\right) = \cos^{-1}\left(\frac{2}{\sqrt{2}\sqrt{3}}\right) = \cos^{-1}\left(\sqrt{\frac{2}{3}}\right) \text{ or } \cos^{-1}\left(\frac{2}{\sqrt{6}}\right) \text{ or } \cos^{-1}\left(\frac{\sqrt{6}}{3}\right) \approx 35.26^\circ$$

A-12, B-12 [§9.5: Equations of Lines and Planes] Let's name our point: $P(1,1,1)$. To construct a vector normal to the plane, first pick any two points on the line $\mathbf{L}(t)$; say $\mathbf{L}(0) = Q(0,2,-1)$ and $\mathbf{L}(1) = R(2,-1,-3)$. Then take a cross product.

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \langle -1,1,-2\rangle \times \langle 1,-2,-4\rangle = \langle -8,-6,1\rangle$$

Now use the vector equation of a plane through the point P.

$$\begin{aligned} \mathbf{n}\bullet(\mathbf{r}-\mathbf{r}_0) &= 0 \\ \langle -8,-6,1\rangle \bullet (\langle x,y,z\rangle - \langle 1,1,1\rangle) &= 0 \\ -8x-6y+z &= -13 \qquad \text{or} \qquad 8x+6y-z = 13 \end{aligned}$$

A-13, B-11 [§9.3: The Dot Product] Pick any point in the plane; say, $C(1,0,0)$. Form the vector from our point $A(2,1,3)$ to C: $\mathbf{v} = \overrightarrow{AC} = \langle -1,-1,-3\rangle$. Read off the normal vector to the plane from the statement of the problem: $\mathbf{n} = \langle 1,-2,1\rangle$.

The object of our desire is either $||\mathbf{proj_n v}||$, the length of the vector projection of $\mathbf{v}$ onto $\mathbf{n}$, or $|\text{comp}_\mathbf{n}\mathbf{v}|$, the absolute value of the scalar projection of $\mathbf{v}$ onto $\mathbf{n}$. Let's compute the latter.

$$\left|\frac{\mathbf{n}\bullet\mathbf{v}}{||\mathbf{n}||}\right| = \left|\frac{-1+2-3}{\sqrt{1+4+1}}\right| = \left|\frac{-2}{\sqrt{6}}\right| = \frac{2}{\sqrt{6}} \quad \text{or } \frac{\sqrt{6}}{3} \approx 0.82$$

A-14, B-16 [§9.8: Arc Length and Curvature]

(a) The curvature formula stated on the exam is in terms of vector functions. So let's embed our parabola (a curve in the xy-plane) in 3-space: $\mathbf{r}(t) = \langle t, 2t^2, 0\rangle$. The osculating circle touches the parabola at the given $\boxed{\text{point}}$ $(0,0) \leftrightarrow P(0,0,0) = \mathbf{r}(0)$.

(b) Compute the radius of curvature $\rho = 1/\kappa(0)$; i.e., the reciprocal of the curvature at P.
With $\mathbf{r}'(t) = \langle 1, 4t, 0\rangle$ and $\mathbf{r}''(t) = \langle 0,4,0\rangle$, we have

$$\kappa(0) = \frac{||\mathbf{r}'(0)\times\mathbf{r}''(0)||}{||\mathbf{r}'(0)||^3} = \frac{||\langle 1,0,0\rangle \times \langle 0,4,0\rangle||}{||\langle 1,0,0\rangle||^3} = \frac{||\langle 0,0,4\rangle||}{||\langle 1,0,0\rangle||^3} = \frac{4}{1} = 4$$

Thus our $\boxed{\text{radius}}$ is $\rho = 1/\kappa(0) = \frac{1}{4}$.

(c) Starting at P, "walk" ρ units in the direction of the unit normal $\mathbf{N}(0)$ to arrive at the osculating circle's center, C.

- *Via clever geometric visualization:* As t increases, the parabola is traversed from left to right. At P, the unit tangent vector points due east; i.e., $\mathbf{T} = \langle 1, 0\rangle$. Accordingly, the unit normal vector points due north; i.e., $\mathbf{N} = \langle 0, 1\rangle$. This is because *the unit normal points toward the concave side of the curve.* Start at P and walk due north (in the direction of $\mathbf{N}$) a radius of $\rho = \frac{1}{4}$ unit; you're now at the $\boxed{\text{center}}$ of the osculating circle, $C(0, \frac{1}{4})$.
- *Via brute force:* Here we'll use Phil Yasskin's method of computing $\mathbf{T}$, $\mathbf{B}$, and $\mathbf{N}$, respectively. ("More is less.") This is actually easier because we already computed the pieces we need during the determination of κ.

$$\begin{aligned}
\mathbf{T}(0) = \mathbf{r}'(0) \ / \ ||\mathbf{r}'(0)|| &= \langle 1,0,0\rangle \ / \ ||\langle 1,0,0\rangle|| = \langle 1,0,0\rangle \\
\mathbf{B}(0) = \mathbf{r}'(0) \times \mathbf{r}''(0) \ / \ ||\mathbf{r}'(0) \times \mathbf{r}''(0)|| &= \langle 0,0,4\rangle \ / \ ||\langle 0,0,4\rangle|| = \langle 0,0,1\rangle \\
\mathbf{N}(0) = \mathbf{B}(0) \times \mathbf{T}(0) &= \langle 0,1,0\rangle \qquad \text{via the right-hand rule!}
\end{aligned}$$

Now proceed as before to get the center at $C(0, \frac{1}{4}, 0) \leftrightarrow (0, \frac{1}{4})$.

(d) Now that you know the center and radius, write down the $\boxed{\text{equation of the osculating circle.}}$

$$(x-0)^2 + (y - \tfrac{1}{4})^2 = (\tfrac{1}{4})^2, \ \ z = 0 \qquad \text{or} \qquad x^2 + y^2 - \tfrac{1}{8}y = 0, \ \ z = 0$$

A-15, B-14 [§9.9: Motion in Space: Velocity and Acceleration]

$$\begin{aligned}
\frac{d\mathbf{v}}{dt} = \mathbf{a}(t) &= \langle 2\cos t, \quad 8\sin t\rangle \\
\mathbf{v}(t) &= \langle 2\sin t, \quad -8\cos t\rangle + \mathbf{C} \\
\langle 2, \ -2\rangle = \mathbf{v}(0) &= \langle 0, \ -8\rangle + \mathbf{C} \quad \Longrightarrow \quad \mathbf{C} = \langle 2, \ 6\rangle \\
\frac{d\mathbf{r}}{dt} = \mathbf{v}(t) &= \langle 2\sin t + 2, \quad 6 - 8\cos t\rangle \\
\mathbf{r}(t) &= \langle 2t - 2\cos t, \quad 6t - 8\sin t\rangle + \mathbf{K} \\
\langle 1, \ 1\rangle = \mathbf{r}(0) &= \langle -2, \ 0\rangle + \mathbf{K} \quad \Longrightarrow \quad \mathbf{K} = \langle 3, \ 1\rangle \\
\mathbf{r}(t) &= \langle 2t + 3 - 2\cos t, \quad 6t + 1 - 8\sin t\rangle
\end{aligned}$$

A-16, B-15 [§10.2: Limits and Continuity] As $(x, y) \to (0, 0)$ along the line $y = x$,

$$\frac{x^3 y}{x^4 + y^4} = \frac{x^4}{2x^4} = \frac{1}{2} \to \frac{1}{2}$$

As $(x, y) \to (0, 0)$ along the line $y = -x$,

$$\frac{x^3 y}{x^4 + y^4} = \frac{-x^4}{2x^4} = -\frac{1}{2} \to -\frac{1}{2}$$

Inasmuch as these directional limits differ, the multivariable limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^4 + y^4}$ does NOT exist.