

Counting and Probability

CONNECTION

Texas Hold'em Poker

In this game of poker a player tries to make the best hand possible using the two cards in the player's hand along with three of the cards on the table.

How is the ranking of poker hands determined? For example, why does a straight flush beat a four-of-a-kind? In this chapter we will learn how to find the probability of a poker hand. See page 101 to find the probability of a flush.



2.1 The Multiplication Principle and Permutations

APPLICATION Number of License Plates

A certain state uses license plates with three letters followed by three numbers with no repeats of letters or numbers. How many such license plates can be made? See Example 10 for the answer.

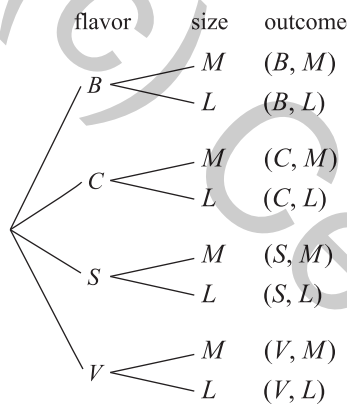


Figure 2.41

◇ The Multiplication Principle

Some of the basic counting techniques that are used in the study of probability will now be considered. The **multiplication principle** given here is fundamental to all of the counting methods that will follow. Before stating the multiplication principle, we consider two examples.

EXAMPLE 1 A Simple Counting Problem A manufacturer makes four flavors of yogurt and each flavor comes in medium and large sizes. The available flavors are blueberry, cherry, strawberry, and vanilla. How many different flavors and sizes of yogurt are there? What is the set of all possible choices?

Solution We can think of the procedure as a sequence of two choices or tasks. The first task is to pick a flavor and the second task is to pick a size. We can denote the choice of picking cherry in the large size as the ordered pair (cherry, large) or simply (C, L). All possible outcomes can be visualized in the **tree diagram** in Figure 2.41. Counting all the possibilities gives eight different yogurts. The set of all possible choices is

$$S = \{(B, M), (B, L), (C, M), (C, L), (S, M), (S, L), (V, M), (V, L)\} \quad \blacklozenge$$

If we examine more closely why we obtained the answer $n(S) = 8$, we can notice from Figure 2.41 that we had four groups of two each. This gives $4 \times 2 = 8$, where there were four choices for flavors, and for each flavor there were two choices for the size. If you are asked to list all of the different outcomes, a tree diagram is essential. However, if you are only asked how many different outcomes there are, the shortcut below is effective.

Rather than list the branches, draw a blank for each place you need to make a choice. Under the blank list what you are choosing. Then determine how many choices you have for each task and multiply, as shown below

$$\frac{\quad}{\text{flavor}} \times \frac{\quad}{\text{size}} = \frac{4}{\text{flavor}} \times \frac{2}{\text{size}} = 8$$

You can abbreviate this shortcut further by simply writing $(_, _)$ to indicate that there are two tasks to be done and then write $(_, _) = (4, 2)$ to show that the first task can be completed four ways and the second task can be completed two ways. Those numbers are multiplied to give eight different outcomes:

$$(_, _) = (4, 2) = 4 \cdot 2 = 8$$

EXAMPLE 2 Forming a Committee A committee of three individuals, Abe, Bertha, and Chia, must pick one of them to be chair and a different one to be

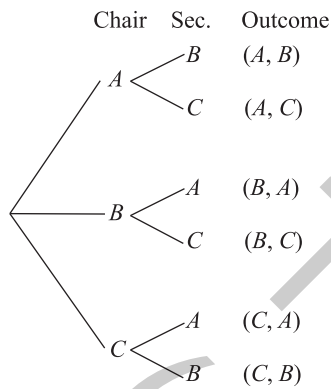


Figure 2.42

secretary. How many ways can this be done? What is the set of all possible outcomes?

Solution The tree diagram in Figure 2.42 breaks this problem into an operation of first picking a chair and then picking a secretary. Notice that there are three choices for chair and, no matter who is picked for chair, there are always two choices for secretary. Thus there are three groups of two each or $3 \times 2 = 6$ possible outcomes. The set of all possible outcomes can be read from the tree diagrams as

$$S = \{(A, B), (A, C), (B, A), (B, C), (C, A), (C, B)\}$$

If the set of all possible outcomes was not needed, the number of outcomes could be found by

$$\frac{\text{chair}}{\text{chair}} \times \frac{\text{secretary}}{\text{secretary}} = \frac{3}{\text{chair}} \times \frac{2}{\text{secretary}} = 6$$

Or simply $(\underline{\quad}, \underline{\quad}) = (3, 2) = 3 \times 2 = 6$

Now suppose there is an operation that consists of a sequence of two tasks with n_1 possible choices for the first task and no matter how the first task was completed, there are always n_2 possible choices for the second task. Figure 2.43 indicates a tree diagram for these two tasks. From this figure we see that there are n_1 groups, each with n_2 elements. Thus there is a total of $n_1 \cdot n_2$ possible outcomes in our sequence of two tasks. This is the **multiplication principle**.

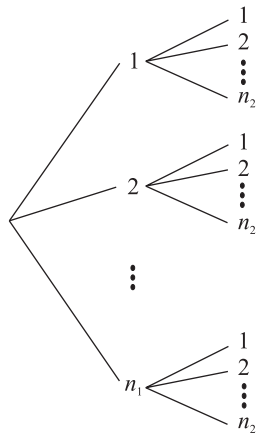


Figure 2.43

Multiplication Principle
 Suppose there is an operation that consists of making a sequence of two tasks with n_1 choices for the first task and, no matter how the first task was completed, there are always n_2 possible choices for the second task. Then there are $n_1 \cdot n_2$ possible ways in which the sequence of two task can be completed.

EXAMPLE 3 Picking Two Horses Ten horses are running in the first race at Aqueduct. You wish to buy a Perfecta ticket, which requires picking the first and second finishers in order. How many possible tickets are there?

Solution You have two tasks: choose the first place horse and choose the second place horse. There are 10 choices for the first task as any of the 10 horses could come in first. Once the horse places first, it is not available to place second, so we have only 9 horses to choose from when we pick a horse for second place. The tasks can be shown as

$$\frac{\text{1st place}}{\text{1st place}} \times \frac{\text{2nd place}}{\text{2nd place}} = \frac{10}{\text{1st place}} \times \frac{9}{\text{2nd place}} = 90$$

Or, this can be summarized as $(\underline{\quad}, \underline{\quad}) = (10, 9) = 10 \cdot 9 = 90$. Thus, by the multiplication principle, there are 90 possible tickets.

We now can see in the same way the general multiplication principle.

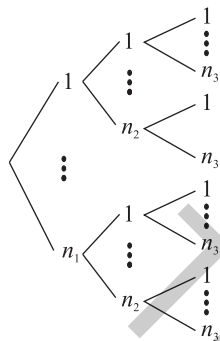


Figure 2.44

General Multiplication Principle

Suppose there is an operation that consists of making a sequence of k tasks with n_1 possible choices for the first task and, no matter what choice was made for the first task, there are always n_2 possible choices for the second task, and, no matter what choices were made for the first two tasks, there are always n_3 possible choices for the third task, and so on. Then there are $n_1 \cdot n_2 \cdots n_k$ possible ways in which the sequence of k tasks can be made.

For example, in Figure 2.44, there are $n_1 \cdot n_2$ elements in the second column, as we noted in Figure 2.43. Emanating from each one of these elements there are n_3 “branches” in the third column. Thus there are $n_1 \cdot n_2 \cdot n_3$ elements in the third column.

EXAMPLE 4 Picking Three Horses Ten horses are running in the first race at Aqueduct. You wish to buy a Trifecta ticket that requires picking the first three finishers in order. How many possible tickets are there?

Solution You must make a choice: (first, second, third). Thus we have a sequence of three blanks (, ,), and we must fill in the blanks. There are 10 choices for first. No matter what horse you pick for first there are always 9 left for second, and no matter which horses you pick for the first and second choices, there are always 8 left for the third. This can be summarized as $(10, 9, 8)$. By the general multiplication principle there are $10 \times 9 \times 8 = 720$ possible tickets. ♦

REMARK: Often the application of the multiplication principle requires some care due to restrictions. Perhaps when you are seating children you wish to have boys and girls alternate. Or when you choose a PIN code you cannot have three of the same digits in the code. Or your computer password must end in two digits. These restrictions require extra care when determining the number of ways a task can be completed.

EXAMPLE 5 Picking Horses In the seventh race at Aqueduct, which is running 10 horses, you have learned through impeccable sources that the overwhelming favorite will be held back from winning. You wish to be assured of having a winning Trifecta ticket. How many must you buy?

Solution Again you must make a choice for each of your three tasks: choose the first, second, and third place horses. This time, from your inside knowledge, you know that there are only nine choices for first, and, given any of these nine choices, there remain nine horses to pick for second. No matter what the first and second picks are, there remains eight choices for the third place horse. This can be summarized as $(9, 9, 8)$. Thus, by the multiplication principle, there are $9 \times 9 \times 8 = 648$ possible choices. Therefore you must buy 648 different tickets to be assured of a winning ticket. ♦

EXAMPLE 6 Forming Words How many three-letter words that all begin with consonants and have exactly one vowel can be made using the first seven letters of the alphabet where using a letter twice is permitted but having two consonants next to each other is not?

Solution Since two consonants cannot be next to each other, one must make the choice (consonant, vowel, consonant). There are five choices for the first consonant in the set $\{b, c, d, f, g\}$. No matter what the first choice there are two choices for vowels $\{a, e\}$, and no matter what the first two choices, there remain the same five consonants to pick from. This can be summarized as $(5, 2, 5)$. Thus the multiplication principle gives $5 \times 2 \times 5 = 50$ possible words. ♦

◆ Factorials

We will be encountering expressions such as $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ and it will be useful to have a symbol to denote this product. We have the following definition.

Factorials

For any natural number n

$$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$$

$$0! = 1$$

Technology Option

See Technology Note 1 on page 89 for details on finding factorials using a graphing calculator or a spreadsheet.

Quick Fact: Factorials can be found on the TI graphing calculator by entering the number and then going pressing **MATH** and choosing the **PRB** menu. The factorial function is fourth option, $4: !$. Select this and then **ENTER** as shown below.

4!	24
5!	120
6!	720
■	

EXAMPLE 7 Calculating Some Factorials Find $4!$, $5!$, and $6!$.

Solution

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

◆ Permutations

The set of digits $\{1, 2, 3, 4\}$ can be arranged in various orders. For example, they can be listed as $(1, 3, 4, 2)$ or perhaps $(4, 3, 1, 2)$. Each of these ordered arrangements differ and we call these different arrangements a **permutation** of the set of digits $\{1, 2, 3, 4\}$. We have the following definition.

Permutations

A **permutation** of a set of elements is an ordered arrangement of all the elements.

By “arrangement” it is understood that elements can be used only once and the order that the elements are in matters. That is the arrangement $\{1, 2\}$ is a different arrangement than $\{2, 1\}$.

EXAMPLE 8 Counting the Number of Permutations Find all permutations of the set $\{a, b, c\}$ and count the total number.

Solution Given the set $\{a, b, c\}$, we can use a tree diagram to find all possible permutations. See Figure 2.45. The set of all possible permutations is

$$S = \{(a, b, c), (a, c, b), (b, a, c), (b, c, a), (c, a, b), (c, b, a)\}$$

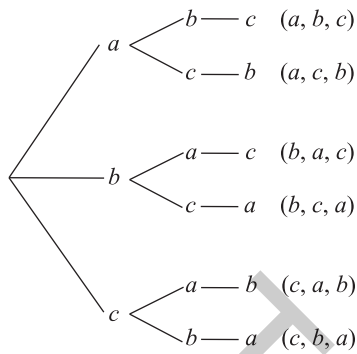


Figure 2.45

There are six possibilities, and these can be seen also by the multiplication principle to be $3 \cdot 2 \cdot 1 = 3!$. ♦

Suppose a set of n distinct objects is given and we wish to find the number of permutations that are possible. We denote this number by $P(n, n)$. There are n choices for the task of choosing which of the n objects is in the first position. No matter which one is chosen first, there always remains $(n - 1)$ objects for the second position. No matter how the first two have been chosen, there remains $(n - 2)$ for the third position, and so on. By the multiplication principle, we then have the total number of possibilities to be

$$P(n, n) = n(n - 1)(n - 2) \cdots 3 \cdot 2 \cdot 1 = n!$$

We thus have the following.

Number of Permutations of n Objects
 The number of permutations of n distinct objects is given by

$$P(n, n) = n(n - 1)(n - 2) \cdots 3 \cdot 2 \cdot 1 = n!$$

EXAMPLE 9 Counting the Number of Permutations In how many ways can a football team of 11 players arrange themselves to trot onto the football field one at a time?

Solution Think of this as arranging 11 different items and then the answer is given by

$$P(11, 11) = 11! = 39,916,800$$

To use the multiplication principle, we have 11 ways to pick which player goes first. He is not available to go second, so there are 10 players (so 10 ways) to complete the task of choosing who runs out second. This continues as follows:

$$\frac{11}{1^{\text{st}} \text{ player}} \times \frac{10}{2^{\text{nd}}} \cdots \frac{2}{10^{\text{th}}} \times \frac{1}{11^{\text{th}} \text{ player}} = 11! = 39,916,800 \quad \blacklozenge$$

There are situations in which we have a set of n distinct objects and wish to select an ordered arrangement of r of them. The total number of such arrangements is denoted by $P(n, r)$. We also refer to this as the **number of permutations of r objects taken from a set of size n objects**. Example 4 was such a case where you selected a first, second, and third place from a set of 10 horses running in a race. Recall from the multiplication principle that we have $P(10, 3) = 10 \cdot 9 \cdot 8$.

We would now like to get a formula for the general case of choosing r items from a set of n items and arranging them. Let's begin by writing our result for $P(10, 3)$ in a different form.

$$\begin{aligned} P(10, 3) &= 10 \cdot 9 \cdot 8 = 10 \cdot 9 \cdot 8 \cdot \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\ &= \frac{10!}{7!} = \frac{10!}{(10 - 3)!} \end{aligned}$$

The generalization of this is left as Exercise 43. We have below the formula for a permutation of r objects taken from a set of n objects.

Technology Option

See Technology Note 2 on page 90 for details on finding permutations using a graphing calculator or a spreadsheet. Quick Fact: permutations can be found on the TI graphing calculator by entering the number n and then going pressing **MATH** and choosing the **PRB** menu. The permutation function is second option, $2:nPr$. Select this and then **ENTER** and then the number n and **ENTER** again.

```
11 nPr 11
39916800
10 nPr 3
720
```



Number of Permutations of r Objects Taken from a Set of Size n

The number of permutations of r distinct objects taken from a set of size n is given by

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

Since $0! = 1$,

$$P(n, n) = \frac{n!}{(n-n)!} = \frac{n!}{1} = n!$$

which is what we obtained earlier.

EXAMPLE 10 License Plates A certain state uses license plates with three letters followed by three digits with no repeats of letters or digits. How many such license plates can be made?

Solution This problem can be done directly from the multiplication principle with six different tasks or as a permutation. Using the multiplication principle we have the following tasks:

$$\frac{26}{\text{letter}} \times \frac{25}{\text{letter}} \times \frac{24}{\text{letter}} \times \frac{10}{\text{digit}} \times \frac{9}{\text{digit}} \times \frac{8}{\text{digit}} = 11,232,000$$

If we use our knowledge of permutations, we can think of deciding on a license plate as a two-task process. The first task is to choose the three letters and then the second task is to choose the three numbers. The number of ways of arranging the three letters without repetitions and when order is important is the number of permutations of 26 letters taken 3 at a time or $P(26, 3)$. The number of ways of arranging the digits without repetitions and when order is important is the number of permutations of 10 digits taken 3 at a time or $P(10, 3)$. By the multiplication principle (when there are two tasks), the total number of different license plates will be

$$P(26, 3)P(10, 3) = (26 \cdot 25 \cdot 24)(10 \cdot 9 \cdot 8) = 11,232,000 \quad \blacklozenge$$

REMARK: If we were not restricted to “no repeats,” there would be $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17,576,000$ different license plates. But, for some states, such as California, that is not enough for each car to have a unique license plate. The current California plates have a digit followed by three letters and three digits which brings the total number of different plates to 175,760,000. States such as Wyoming have a county code (from 1 to 23) followed by three letters. There are $23 \cdot 26 \cdot 26 \cdot 26 = 404,248$ possible plates using that pattern.

EXAMPLE 11 Movie Seatings Matthew and Jennifer go to the movies with four of their friends. How many ways can these six children be seated if

- there are no restrictions?
- Matthew and Jennifer are seated next to each other?
- Matthew and Jennifer are not next to each other?

Solution

- a. There are six children to be seated, so we can look at this as a sequence of six tasks, (first, second, third, fourth, fifth, sixth) = $(6, 5, 4, 3, 2, 1) = 6!$ or we can recognize this as a permutation of six objects taken 6 at a time to find $P(6,6) = 6!$. Either way the number of ways for the children to be seated is 720.
- b. There is more than one way to assure that Matthew and Jennifer are seated together. One way is to “glue” Matthew and Jennifer together. That is, we will treat them as a single object in the initial seating arrangement. Then we will have two tasks. The first task will be to arrange the five objects (four children and the object that is Matthew and Jennifer glued together). The next task is to “un-glue” Matthew and Jennifer and determine how many ways those two objects can be arranged.

$$\frac{\text{arrange 5 objects}}{\text{arrange 5 objects}} \times \frac{\text{arrange M and J}}{\text{arrange M and J}} = \frac{5!}{\text{arrange 5 objects}} \times \frac{2!}{\text{arrange M and J}} = 120 \cdot 2 = 240$$

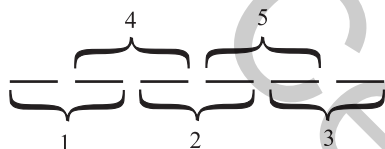


Figure 2.46

Or, we could think of it as a series of three tasks where we begin by finding how many pairs of seats Matthew and Jennifer could sit in. The next task is to determine how many ways Matthew and Jennifer could sit in their pair of seats and the last task is to seat the rest of the children. If we look at Figure 2.46, we see there are five pairs of seats for Matthew and Jennifer. There will be $2!$ ways that Matthew and Jennifer can sit in the pair of seats. For the third task there are four empty seats and four children to be seated, so there will be $4!$ ways to complete this task. In all we have

$$\frac{5}{\text{choose pair of seats}} \times \frac{2!}{\text{arrange M and J}} \times \frac{4!}{\text{arrange remaining children}} = 5 \cdot 2 \cdot 24 = 240$$

- c. In the 720 ways to seat the children found in the first part, Matthew and Jennifer are seated together or they are not seated together. These are mutually exclusive events as they can't be seated together and not seated together at the same time. So, since there are 240 ways they are seated together, there must be $720 - 240 = 480$ ways they are not seated together. ♦

◆ Technology Corner

🌀 Technology Note 1 Factorials

The commands for many counting operations are found by pressing the **MATH** button and arrowing over to the **PRB** menu as shown in Screen 1. To evaluate a factorial, enter the number first and then **MATH** then **PRB** and finally $4:!$. This returns you to the homepage where **ENTER** will evaluate the factorial. See the note on page 86 to see this function in use.

To find factorials using a spreadsheet such as Excel, place your cursor in a cell and type `=FACT(` followed by the value and a closing `)`. Enter and the factorial will be evaluated. See Worksheet 1. Alternatively, you can put the number to be evaluated in a cell and then refer to that cell in the **FACT** function.

```

MATH NUM CPX 17:35
1:rand
2:nPr
3:nCr
4:!
5:randInt(
6:randNorm(
7:randBin(

```

Screen 1

	A	B
1	=FACT(8)	
2	6	=FACT(A2)

	A	B
1	40320	
2	6	720

Worksheet 1

	A	B	C
1	=PERMUT(8,3)		
2	6	4	=PERMUT(A2,B2)

	A	B	C
1		336	
2		6	4
			360

Worksheet 2

Technology Note 2 Permutations

The calculator command for permutations is accessed via the PRB menu shown in Screen 1. To use it, enter then number of objects, n then press **MATH**, then arrow over to PRB and select 2:nPr. Press **ENTER** then the number selected, r and then **ENTER**. See the note on page 88 to see this function in use.

To find permutations using a spreadsheet such as Excel, place your cursor in a cell and enter =PERMUT(. Then enter the number of objects followed by a comma and the number selected. Close with a) and enter. See Worksheet 2. Alternatively, enter the number of objects and the number chosen in two cells and refer to these cells in the PERMUT command.

Self-Help Exercises 2.1

- A restaurant serves 3 soups, 4 salads, 10 main dishes, and 6 desserts. How many different meals can be served if one of each category is chosen?
- Six junior executives and three senior executives are to line up for a picture. The senior executives must all be lined up together on the left and the junior executives must be all together on the right. In how many ways can this be done?
- Four couples are going to the movie together. How many ways can these eight people be seated if couples sit together?

2.1 Exercises

In Exercises 1 through 12, evaluate the given expression.

- $P(5, 3)$
- $P(5, 2)$
- $P(8, 5)$
- $P(8, 3)$
- $P(7, 7)$
- $P(7, 1)$
- $P(9, 9)$
- $P(9, 1)$
- $P(9, 2)$
- $P(n, 0)$
- $P(n, 1)$
- $P(n, 2)$
- A manufacturer offers 4 styles of sofas with 30 fabrics for each sofa. How many different sofas are there?
- A manufacturer offers 3 grades of carpet with 10

colors for each grade. How many different carpets are there?

- A restaurant offers 4 types of salads, 10 main dishes, and 5 desserts. How many different complete meals are there?
- A card is picked from a standard deck of 52, and then a coin is flipped twice. How many possible outcomes are there?
- A state's license plates have six digits with repetitions permitted. How many possible license plates of such type are there?

18. A state's license plates have five letters with repetitions not permitted. How many possible license plates of such type are there?
19. An automobile manufacturer offers a certain style car with two types of radios, 10 choices of exterior colors, 5 different interior colors, and 3 types of engines. How many different automobiles are offered?
20. A contractor has four styles of homes, each with three styles of garages, four styles of decks, and five styles of carpeting. How many possibilities are there?
21. A state makes license plates with three letters followed by three digits with repetitions permitted. How many possibilities are there?
22. A state makes license plates with three letters followed by three numbers with no repetitions of letters permitted. How many possibilities are there?
23. How many three-letter code words can be made from the first eight letters of the alphabet if consonants cannot be next to each other and letters cannot be repeated?
24. How many five-letter code words can be made from the first eight letters of the alphabet if consonants cannot be next to each other, vowels cannot be next to each other, and vowels cannot be repeated but consonants can be?
25. At an awards ceremony, five men and four women are to be called one at a time to receive an award. In how many ways can this be done if men and women must alternate?
26. At an awards ceremony, five women and four men are each to receive one award and are to be presented their award one at a time. Two of the awards are to be first given to two of the women, and then the remaining awards will alternate between men and women. How many ways can this be done?
27. In how many ways can the five members of a basketball team line up in a row for a picture?
28. In how many ways can the individuals in a foursome of golfers tee off in succession?
29. An executive is scheduling meetings with 12 people in succession. The first two meetings must be with two directors on the board, the second four with four vice presidents, and the last six with six junior executives. How many ways can this schedule be made out?
30. An executive is scheduling trips to the company's European plants. First the four French plants will be visited, followed by the three Italian plants, and then the five German ones. How many ways can this schedule be made out?
31. The starting nine players on the school baseball team and the starting five players on the basketball team are to line up for a picture with all members of the baseball team together on the left. How many ways can this be done?
32. The seven starting offensive linemen and four starting offensive backs of the New York Giants are to line up for a picture with the seven linemen in the middle. How many ways can this be done?
33. On a baseball team, the three outfielders can play any of the three outfield positions, and the four infielders can play any of the four infield positions. How many different arrangements of these seven players can be made?
34. On a football team, the seven linemen can play any of the seven linemen positions and the four backs can play any of the backfield positions. How many different arrangements of these 11 players can be made?
35. A group of 12 must select a president, a vice president, a treasurer, and a secretary. How many ways can this be done?
36. In the Superfecta, one must pick the first four finishers of a horse race in correct order. If there are 10 horses running in a race, how many different tickets are there?
37. A buyer for a furniture store selects 8 different style sofas from a group of 10 and has each style shipped on successive weeks. How many ways can this be done?
38. A chef can make 12 main courses. Every day a menu is formed by selecting 7 of the main courses and listing them in order. How many different such menus can be made?
39. Two groups are formed with 10 in the first group and 8 different people in the second. A president, vice president, and a secretary/treasurer is to be chosen in each group. How many ways can this be done?

40. At a race track you have the opportunity to buy a ticket that requires you to pick the first and second place horse in the first two races. If the first race runs 8 horses and the second runs 10, how many different tickets are possible?
41. A picture is to be taken by lining up 4 of the 11 players from the football team on the left, then 3 of the 9 players from the baseball team in the center, and finally 2 of the 5 players from the basketball team on the right. How many ways can this arrangement be done?
42. A tourist has eight cities in Great Britain, six in France, five in Italy, and seven in Germany on a list she would like to visit. She decides that she will

first go to Great Britain and visit four of the cities on her list, then on to France to visit three cities on the list, then on to Italy for two cities, and then on to Germany to visit four on the list. How many ways can her itinerary be made out?

Extensions

43. Prove the general case that $P(n, r) = \frac{n!}{(n-r)!}$
44. How many different ways can seven people be seated at a round table with seven seats?
45. How many different ways can five condiments be placed along the edge of a lazy susan (a round tray that spins)?

Solutions to Self-Help Exercises 2.1

1. A restaurant that serves 3 soups, 4 salads, 10 main dishes, and 6 desserts can by the multiplication principle serve $3 \times 4 \times 10 \times 6 = 720$ different meals.
2. Six junior executives can line up in $6!$ ways on the right. For each of these ways the three senior executives can line up in $3!$ ways on the left. Thus by the multiplication principle, the two groups can line up in $3! \times 6! = 4320$ ways.
3. Consider each couple to be a single object. We have $4!$ ways to arrange the four couples. Each couple can be seated in $2!$ ways so in all there are

$$\frac{4!}{\text{arrange couples}} \times \frac{2!}{\text{couple 1}} \times \frac{2!}{\text{couple 2}} \times \frac{2!}{\text{couple 3}} \times \frac{2!}{\text{couple 4}} = 24 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 384$$

2.2 Combinations

APPLICATION

Connecticut Lotto Game

In the state of Connecticut's lotto game, 6 numbered balls are randomly selected without replacement from a set of 44 to determine a winning set of numbers (without regard to order). If no one picks these 6 numbers, the money wagered stays in the "pot" for the next drawing. After several consecutive games with no winners, the pot gets large and attracts a lot of attention and ticket sales.

When the Connecticut lotto game started, the total number of balls was 36. After a number of years when there were very few long streaks with no winners, lotto officials changed the total number of balls from 36 to 44. Suppose a number of weeks has gone by without a winner and the pot has grown very large. You wish to organize a syndicate of investors that will purchase every possible ticket to ensure obtaining a winning ticket. How many tickets will the syndicate have to buy if there are 36 balls? If there are 44? The answer can be found in the discussion before Example 1.

◇ Combinations

When a permutation of n distinct objects is taken r at a time, one selects r of the objects in a specific order. A combination of r distinct objects taken from a set of size n is merely a selection of r of the objects (without concern for order). We consider combinations in this section.

Given the set $\{a, b, c\}$ we know from the last section that there are $P(3, 2) = 3 \times 2 = 6$ ways of selecting two of these at a time when order is important. The six ways are

$$(a, b), (b, a), (a, c), (c, a), (b, c), (c, b)$$

If now we wish to select two at a time when order is not important, then (a, b) is the same as (b, a) , and (a, c) is the same as (c, a) , and (b, c) is the same as (c, b) . Thus there are only three ways of selecting two objects at a time when order is not important: $\{a, b\}$, $\{a, c\}$, $\{b, c\}$. These are referred to as **combinations**.

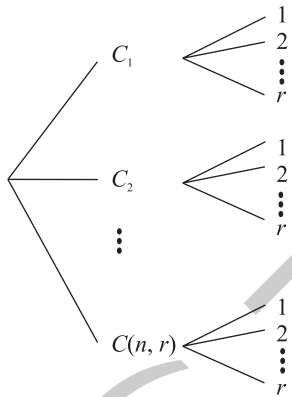
Combinations

A **combination** of r distinct objects taken from a set of size n is a selection of r of the objects (without concern for order).

For example, if from a group of four people, we wish to select a president, a vice president, and a secretary/treasurer, then order is important. If, on the other hand, we wish to select a committee of three people from a set of four, then order is not important since the duties and title of each committee member is the same no matter in what order they are selected.

We know from the last section that the number of permutations of r objects taken from a set of size n is given by $P(n, r)$. We denote the number of combinations of r objects taken from a set of size n by $C(n, r)$. We wish to find a formula for $C(n, r)$.

We can see this by viewing the process of selecting all permutations of r objects taken from a set of size n as a sequence of two tasks.



1. The first task is to select r distinct objects where the order that the objects are chosen doesn't matter. The number of ways to complete this task will be $C(n, r)$ as this is the definition of what we mean by combination.
2. In the second task the r objects selected in the combination are arranged in some order. The number of ways to complete this task is $r!$, as we learned in the last section. Notice that no matter what combination we take, we then always permute by the same number, $(r!)$.

That is, first select a combination of r distinct objects and then order them. By the multiplication principle we have

$$P(n, r) = C(n, r)r! \quad \rightarrow \quad C(n, r) = \frac{1}{r!}P(n, r) = \frac{n!}{(r!)(n-r)!}$$

See Figure 2.47 where the distinct combinations are listed as $C_1, C_2, \dots, C(n, r)$. We have proven the following.

Figure 2.47

Technology Option

See Technology Note 1 on page 97 for details on finding combinations using a spreadsheet. Combinations can be found on the TI graphing calculator by entering the number and then going pressing **MATH** and choosing the **PRB** menu. The combination function is third option, $3:nCr$. Select this and then **ENTER** followed by r and another **ENTER** as shown below.

```
36 nCr 6 1947792
44 nCr 6 7059052
52 nCr 5 2598960
```

Number of Combinations of r Objects taken from a Set of Size n

The number of combinations of r distinct objects taken from a set of size n , denoted by $C(n, r)$, is given by

$$C(n, r) = \frac{n!}{(r!)(n-r)!}$$

We now solve the problem posed at the beginning of this chapter. Since the order in which the numbers are selected is of no consequence, we are looking for $C(36, 6)$ in the first case and $C(44, 6)$ in the second case. These are

$$C(36, 6) = \frac{36!}{6!(36-6)!} = \frac{36 \cdot 35 \cdot 34 \cdot 33 \cdot 32 \cdot 31 \cdot (30!)}{6!(30!)} = \frac{36 \cdot 35 \cdot 34 \cdot 33 \cdot 32 \cdot 31}{6!} = 1,947,792$$

and

$$C(44, 6) = \frac{44!}{6!(44-6)!} = \frac{44 \cdot 43 \cdot 42 \cdot 41 \cdot 40 \cdot 39 \cdot (38!)}{6!(38!)} = \frac{44 \cdot 43 \cdot 42 \cdot 41 \cdot 40 \cdot 39}{6!} = 7,059,053$$

Thus with 36 balls, the syndicate must buy about 2 million tickets. If there are 44 balls, then about 7 million tickets must be bought to guarantee a winning ticket.

EXAMPLE 1 Calculating the Number of Different Poker Hands How many different 5-card poker hands can be dealt from a standard deck of 52 cards?

Solution Refer to Figure 2.48 for a standard deck of cards. Since the order in which the cards in a poker hand are dealt is of no consequence, we are looking for $C(52, 5)$ which is

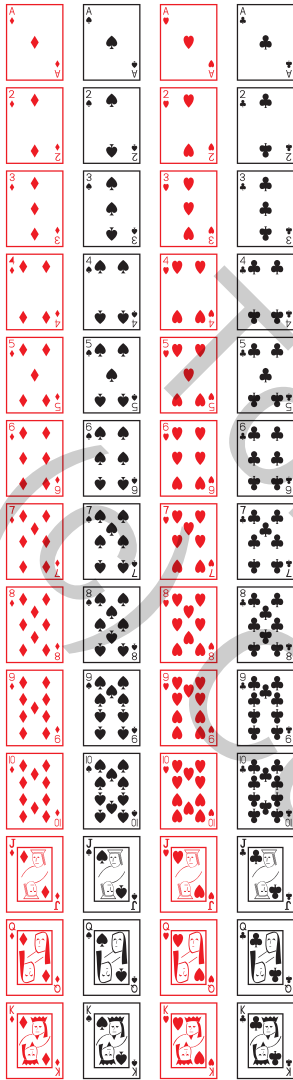


Figure 2.48

$$C(52, 5) = \frac{52!}{5!(52 - 5)!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot (47!)}{5!(47!)} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5!} = 2,598,960$$

EXAMPLE 2 A Counting Problem in Poker Find the number of poker hands with three queens and two jacks.

Solution View this as a sequence of two tasks. In the first task, select three queens from a deck with four queens. In the second task, select two jacks from a deck with four jacks. Since order is not important in either case, the first task can be done in $C(4, 3)$ ways and the second task can be done in $C(4, 2)$ ways. No matter how the first task is completed, the second task can always be made in $C(4, 2)$ ways, the multiplication principle then indicates that the number of hands with three queens and two jacks is given by

$$C(4, 3)C(4, 2) = \left(\frac{4!}{3!(4 - 3)!}\right) \left(\frac{4!}{2!(4 - 2)!}\right) = \left(\frac{4(3!)}{3!}\right) \left(\frac{4 \cdot 3 \cdot 2}{2 \cdot 2}\right) = (4)(6) = 24$$

EXAMPLE 3 Counting Using Both Combinations and Permutations In how many ways can a committee be formed with a chair, a vice chair, a secretary/treasurer, and 4 additional people all chosen from a group of 10 people?

Solution This example, like many other counting problems, can be solved in more than one way. We illustrate four ways this question can be solved. Of course all of them lead to the same correct value!

- Using the multiplication principle we have a series of tasks. Our first task is to choose a chair of the committee. We will have 10 choices for this task. The person chosen cannot be chosen again for vice chair, so we will have 9 ways to complete the task of choosing the vice chair. That leaves us with 8 ways to choose the secretary/treasurer. Now there are 7 people available to be put in the remaining spots on the committee and the order that these are chosen doesn't matter. Therefore there will be $C(7, 4)$ ways to complete the task of choosing the rest of the committee. This is summarized below:

$$\frac{10}{\text{chair}} \times \frac{9}{\text{vicechair}} \times \frac{8}{\text{S/T}} \times \frac{C(7,4)}{\text{rest}} = 10 \cdot 9 \cdot 8 \cdot 35 = 25,200$$

- Another way of doing this problem is to see the process as a sequence of two selections. The first is to pick the chair, vice chair, and secretary/treasurer from the set of 10. Since order is important, this can be done in $P(10, 3)$ ways. The second selection is to then pick the other 4 members of the committee from the remaining 7 people. This can be done in $C(7, 4)$ ways since the order is not important. Furthermore, since the second selection can always be made in $C(7, 4)$ ways no matter how the first selection was made, the multiplication principle indicates that the total number of ways of selecting the committee is

$$P(10, 3)C(7, 4) = 10 \cdot 9 \cdot 8 \frac{7 \cdot 6 \cdot 5 \cdot 4}{4 \cdot 3 \cdot 2} = 25,200$$

3. Yet another way of viewing this is to select four of the committee members from the group of 10 and then select the chair, vice chair, and secretary/treasurer from the remaining six. This gives

$$C(10,4)P(6,3) = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2} 6 \cdot 5 \cdot 4 = 25,200$$

4. Finally, another way of looking at this is to select the committee of seven from the group of 10 and then select the chair, vice chair, and the secretary/treasurer from the committee of seven. This gives

$$C(10,7)P(7,3) = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} 7 \cdot 6 \cdot 5 = 25,200 \quad \blacklozenge$$

EXAMPLE 4 A Counting Problem Using Combinations A committee of 15 people consists of eight men and seven women. In how many ways can a subcommittee of five be formed if the subcommittee consists of

- any five committee members?
- all men?
- at least three men?

Solution

- a. Since the order of selection is of no consequence, the answer is

$$\begin{aligned} C(15,5) &= \frac{15!}{5!(15-5)!} = \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot (10!)}{5!(10!)} = \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11}{5!} \\ &= 3003 \end{aligned}$$

- b. Here we must pick five men out of eight possible men. This can be done in $C(8,5)$ ways since again order is not important. Thus

$$C(8,5) = \frac{8!}{5!(8-5)!} = \frac{8 \cdot 7 \cdot 6 \cdot (5!)}{5!(3 \cdot 2)} = 56$$

- c. At least three men means all subcommittees with three men and two women, plus all subcommittees with four men and one women, plus all with five men. This is

$$\begin{aligned} &C(8,3)C(7,2) + C(8,4)C(7,1) + C(8,5)C(7,0) \\ &= \frac{8 \cdot 7 \cdot 6}{3 \cdot 2} \binom{7 \cdot 6}{2} + \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2} (7) + \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{5 \cdot 4 \cdot 3 \cdot 2} (1) \\ &= (56)(21) + (70)(7) + (56)(1) = 1722 \quad \blacklozenge \end{aligned}$$

◆ **Additional Applications of Combinations**

EXAMPLE 5 A Counting Problem Involving a Sequence An investor has selected a growth mutual fund from a large set of growth funds and will consider any of the next 10 years a success (S) if this mutual fund performs above average in the set of funds and a failure (F) otherwise.

- How many different outcomes are possible?

- b. How many different outcomes have exactly six successes?
 c. How many different outcomes have at least three successes?

Solution

- a. An outcome consists of 10 operations in sequence. Each operation assigns a S or F . One such example is

$$(S, S, F, S, S, F, F, F, S, S)$$

No matter what the assignments of S 's and F 's in any of the prior years, there are always two possibilities for the current year: S or F . Using the multiplication principle with $k = 10$ and $n_1 = n_2 = \dots = n_{10} = 2$ yields $2^{10} = 1024$ as the total number of possible outcomes.

- b. An outcome with exactly 6 S 's was given in the first part. Notice that this amounts to filling in six years with S 's and four years with F 's. A particular outcome will be determined once we fill in 6 S 's in six of the years. This can be done in $C(10, 6)$ ways. This is

$$C(10, 6) = \frac{10!}{6!(10-6)!} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2} = 210$$

- c. The answer to this is the number with exactly 3 S 's plus the number with exactly 4 S 's plus ... the number with exactly 10 S 's, or

$$C(10, 3) + C(10, 4) + \dots + C(10, 10).$$

A shorter way can be given by noticing that this is just the total number less the number with at most 2 S 's. This is

$$1024 - [C(10, 0) + C(10, 1) + C(10, 2)] = 1024 - [1 + 10 + 45] = 968. \quad \blacklozenge$$

EXAMPLE 6 Selecting a Jury Twenty people are called for jury duty. A jury of 12 will be selected at random followed by the selection of two alternate jurors. How many ways can this be done?

Solution This is a two-step process. We begin by selecting the jury where 12 are chosen from 20 and order does not matter. This can be done $C(20, 12)$ ways. The second task is to choose the alternate jurors from the remaining $20 - 12 = 8$ people. This can be done $C(8, 2)$ ways. In all

$$\frac{C(20, 12)}{\text{jury}} \times \frac{C(8, 2)}{\text{alternates}} = 125970 \cdot 28 = 35,271,600$$

There are 35,271,600 different ways that the jury and the alternates can be chosen. \blacklozenge

\blacklozenge **Technology Corner**

 **Technology Note 1 Combinations**

To find combinations using a spreadsheet such as Excel, place your cursor in a cell and enter =COMBIN(. Then enter the number of objects followed by a

	A	B	C
1	1947792		
2	44	6	7059052

	A	B	C
1	=COMBIN(36,6)		
2	44	6	=COMBIN(A2,B2)

comma and the number selected. Close with a) and enter. See Worksheet 3. Alternatively, enter the number of objects and the number chosen in two cells and refer to these cells in the COMBIN command.

Worksheet 3

Self-Help Exercises 2.2

1. A quinella ticket at a race track allows one to pick the first two finishers without regard to order. How many different tickets are possible in a race with 10 horses?
2. A company wishes to select 4 junior executives

from the San Francisco office, 5 from the Dallas office, and 5 from the Miami office to bring to the New York City headquarters. In how many ways can this be done if there are 10 junior executives in San Francisco, 12 in Dallas, and 15 in Miami?

2.2 Exercises

In Exercises 1 through 12, calculate the indicated quantity.

1. $C(8, 3)$
2. $C(8, 4)$
3. $C(8, 5)$
4. $C(12, 12)$
5. $C(12, 1)$
6. $C(12, 0)$
7. $C(7, 4)$
8. $C(7, 3)$
9. $C(15, 2)$
10. $C(n, 0)$
11. $C(n, 1)$
12. $C(n, 2)$
13. Find all permutations of $\{a, b, c\}$ taken two at a time by first finding all combinations of the set taken two at a time and then permuting each combination. Construct a tree similar to Figure 2.47 in the text.
14. Find all permutations of $\{a, b, c, d\}$ taken three at a time by first finding all combinations of the set taken three at a time and then permuting each combination. Construct a tree similar to Figure 2.47 in the text.
15. If you have a penny, a nickel, a dime, a quarter, and a half-dollar in your pocket or purse, how many different tips can you leave using 3 coins?
16. From a list of 40 captains, 5 are to be promoted to major. In how many ways can this be done?
17. In a certain lotto game, six numbered balls are randomly selected without replacement from a set of balls numbered from 1 to 46 to determine a winning set of numbers (without regard to order). Find the number of possible outcomes.

18. A boxed Trifecta ticket at a horse track allows you to pick the first three finishers without regard to order. How many different tickets are possible in a race with 10 horses?
19. From a list of 20 recommended stocks from your brokerage firm, you wish to select 5 of them for purchase. In how many ways can you do this?
20. A restaurant offers eight toppings on its pizza. In how many ways can you select three of them?
21. If you join a book club, you can purchase 4 books at a sharp discount from a list of 20. In how many ways can you do this?
22. A firm is considering expanding into four of nine possible cities. In how many ways can this be done?
23. In how many ways can an inspector select 5 bolts from a batch of 40 for inspection?
24. A chef has 20 dinners that she can make. In how many ways can she select 6 of them for the menu for today?
25. In her last semester, a student must pick 3 mathematics courses and 2 computer science courses to graduate with a degree in mathematics with a minor in computer science. If there are 11 mathematics courses and 7 computer science courses available to take, how many different ways can this be done?

26. A committee of 12 U.S. senators is to be formed with 7 Democrats and 5 Republicans. In how many ways can this be done if there are 53 Democratic senators and 47 Republican senators?
27. A chef has 20 main courses and six soups that he can prepare. How many different menus could he require if he always has seven main courses and three soups on each menu?
28. A firm must select 4 out of a possible 10 sites on the East Coast and 3 out of a possible 8 sites on the West Coast for expansion. In how many ways can this be done?
29. A firm has 12 junior executives. Three are to be sent to Pittsburgh, one to Houston, one to Atlanta, and one to Boston. In how many ways can this be done?
30. Six prizes are to be given to six different people in a group of nine. In how many ways can a first prize, a second prize, a third prize, and three fourth prizes be given?
31. In a new group of 11 employees 4 are to be assigned to production, 1 to sales, and 1 to advertising. In how many ways can this be done?
32. A parent of seven children wants two children to make dinner, one to dust, and one to vacuum. In how many ways can this be done?
33. In how many ways can the nine member Supreme Court give a five-to-four decision upholding a lower court?
34. In how many ways can a committee of five reach a majority decision if there are no abstentions?
35. A coin is flipped eight times in succession. In how many ways can exactly five heads occur?
36. A coin is flipped eight times in succession. In how many ways can at least six heads occur?
37. A coin is flipped eight times in succession. In how many ways can at least two heads occur?
38. A baseball team takes a road trip and plays 12 games. In how many ways could they win 7 and lose 5?
39. A banana split is made with 3 scoops of ice cream, three different flavors of ice cream, three different syrups, two different types of nuts, and with or without whipped cream. How many different banana splits can be made if there are twelve flavors of ice cream, eight syrups, and four types of nuts to choose from?
40. A salesman has 10 customers in New York City, 8 in Dallas, and 6 in Denver. In how many ways can he see 4 customers in New York City, 3 in Dallas, and 4 in Denver?
41. Find the number of different poker hands that contain exactly three aces, while the remaining two cards do not form a pair.
42. Find the number of full houses in a poker hand, that is, the number of poker hands with three of a kind and two of a kind.
43. Find the number of poker hands with two pairs, that is, two different two of a kinds with the fifth card a third different kind.
44. In how many ways can a doubles game of tennis be arranged from eight boys and four girls if each side must have one boy and one girl?
45. Show that $C(n, r) = C(n, (n - r))$.

Solutions to Self-Help Exercises 2.2

Technology Option

```

10 nCr 2          45
(10 nCr 4)*(12 n
Cr 5)*(15 nCr 5)
499458960

```

1. The number of tickets is the same as the number of ways of selecting 2 objects from 10 when order is not important. That is

$$C(10, 2) = \frac{10 \cdot 9}{2} = 45$$

2. The number of ways that junior executives can be selected from San Francisco, Dallas, and Miami, is respectively, $C(10, 4)$, $C(12, 5)$, and $C(15, 5)$. By the multiplication principle the total number of ways this can be done is

$$\begin{aligned}
C(10,4)C(12,5)C(15,5) &= \frac{10!}{4!(10-4)!} \frac{12!}{5!(12-5)!} \frac{15!}{5!(15-5)!} \\
&= \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2} \cdot \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{5 \cdot 4 \cdot 3 \cdot 2} \cdot \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11}{5 \cdot 4 \cdot 3 \cdot 2} \\
&= (10 \cdot 3 \cdot 7)(11 \cdot 9 \cdot 8)(7 \cdot 13 \cdot 3 \cdot 11) \\
&= 499,458,960
\end{aligned}$$

2.3 Probability Applications of Counting Principles

APPLICATION Probability in Games

Find the probability of drawing a flush, but not a straight flush, in a poker game, assuming that any 5-card hand is just as likely as any other. For the answer see Example 4.

Recall from the previous chapter that the probability of an event E is given by

$$P(E) = \frac{n(E)}{n(S)}$$

where $n(E)$ is the number of outcomes in event E and $n(S)$ is the number of outcomes in the uniform sample space for this experiment. Up to this point, $n(S)$ was found using a tree diagram or simply listing the outcomes of the experiment. In this section the sample spaces will be found using the counting techniques learned in this chapter.

EXAMPLE 1 Two Defective Transistors A bin contains 15 identical (to the eye) transistors except that 6 are defective and 9 are not. Suppose a transistor is selected from the bin and then another is selected without replacing the first. What is the probability that both transistors are defective?

Solution First note that we wish to select 2 transistors from 15 with order not important. The number of ways to do this is

$$C(15,2) = \frac{15 \cdot 14}{2} = 105$$

The number of ways we can select 2 defective transistors from a set of 6 is

$$C(6,2) = \frac{6 \cdot 5}{2} = 15$$

If the event E is “both transistors are defective,” then

$$P(E) = \frac{C(6,2)}{C(15,2)} = \frac{15}{105} = \frac{1}{7} \quad \blacklozenge$$

EXAMPLE 2 Defective Transistors Again What is the probability of selecting 5 transistors from the bin in Example 1 with 2 defective and 3 not defective?

Solution The number of ways of selecting 5 from the 15 is

$$C(15,5) = \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11}{5 \cdot 4 \cdot 3 \cdot 2} = 3003$$

The number of ways of selecting 2 defectives from 6 is $C(6,2)$, while the number of ways of selecting 3 non-defective ones from 9 is $C(9,3)$. Thus by the multiplication principle, the number of ways of doing both is

$$C(6,2) \cdot C(9,3) = \frac{6 \cdot 5}{2} \cdot \frac{9 \cdot 8 \cdot 7}{3 \cdot 2} = 1260$$

If E is the probability of selecting 2 defective transistors and 3 non-defective ones, then

$$P(E) = \frac{1260}{3003} \approx 0.420 \quad \blacklozenge$$

EXAMPLE 3 Coin Tosses A fair coin is tossed six times. Assuming that any outcome is as likely as any other, find the probability of obtaining exactly three heads.

Solution By the multiplication principle, there are $2^6 = 64$ possible outcomes. The number of ways of obtaining exactly three heads is the number of ways of selecting three slots from among six to place the heads. This is

$$C(6,3) = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2} = 20$$

Thus if E is the event that exactly three heads occur, then

$$P(E) = \frac{20}{64} = \frac{5}{16} \quad \blacklozenge$$

EXAMPLE 4 A Poker Hand Find the probability of drawing a flush, but not a straight flush, in a poker game, assuming that any 5-card hand is just as likely as any other.

Solution A deck in poker has 52 cards. Thus there are

$$C(52,5) = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2} = 2,598,960$$

possible hands.

A flush consists of 5 cards in a single suit. There are four suits, each of 13 cards. Thus the number of ways of obtaining a flush in a particular suit is

$$C(13,5) = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5 \cdot 4 \cdot 3 \cdot 2} = 1287$$

A straight consists of 5 cards in sequence. There are 10 such straight flushes in each suit:

$$\{1, 2, 3, 4, 5\}, \dots, \{10, J, Q, K, A\}$$

Thus the number of flushes that are not straights is $4(1287 - 10) = 5108$. Therefore if E is the event of drawing a flush, but not a straight flush, then

$$P(E) = \frac{5108}{2,598,960} \approx 0.0020 \quad \blacklozenge$$

REMARK: The ranking of poker hands depends on the difficulty (probability) of obtaining that hand. In the exercises the probabilities are calculated and this will demonstrate why, for example, a straight flush is a higher ranked hand than four-of-a-kind.

◇ Distinguishable Permutations

In our previous counting problems we have only arranged items that were different. That is, if we arranged the letters in the word *ERG* we would have six ways to arrange them and they are all different:

ERG EGR REG GER GRE RGE

If we try to do the same with the word *EGG* notice what happens

EGG EGG GEG GEG GGE GGE

It appears there are duplicate arrangements. This is because while there are two *G*'s, they are identical. So if we want to only count those arrangements that look different, it seems we must take into account the fact that some of the items we are arranging are identical. A similar case occurs when partitioning items into multiple groups. The general formula for dealing with these cases will be developed in the following examples.

EXAMPLE 5 **Dividing a Committee to Perform Tasks** In how many ways can a group of 14 people be divided into three committees, each assigned a different task, the first committee consisting of 8 people, the second 4 people, and the third 2 people?

Solution We can select 8 people for the first committee in $C(14, 8)$ ways and then select 4 people for the second committee from among the remaining 6 people in $C(6, 4)$ ways. The remaining 2 people can then go into the third committee in only one way. By the multiplication principle, the total number of ways this can be done is

$$C(14, 8) \cdot C(6, 4) = \frac{14!}{8!6!} \cdot \frac{6!}{4!2!} = \frac{14!}{8!4!2!}$$

We leave the answer in this form in view of what we are about to do. However, the actual value is 45,045. ◇

We can think of the three committees in the previous problem as a division of the set of $n = 14$ people into three groups. We select n_1 elements from the set of n . Now select n_2 elements from the remaining $n - n_1$ elements. Finally, place the remaining n_3 elements in the third group. The total number of ways this can be done is then, in analogy with the committee of 14 discussed above,

$$\frac{n!}{n_1!n_2!n_3!}$$

where $n_1 + n_2 + n_3 = n$.

REMARK: You can do this as an exercise by realizing that the answer should be $C(n, n_1)C(n - n_1, n_2)$. Now compute this and obtain the answer given above.

Distinguishable Permutations

A set of size n is divided into k groups of sizes n_1, n_2, \dots, n_k , with all elements in each group being identical. Then the number of distinguishable permutations is

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

where $n_1 + n_2 + \cdots + n_k = n$.

EXAMPLE 6 Arrangements of Letters Find the number of arrangements of the word MISSISSIPPI.

Solution We can think of this as forming the S-committee (or S-group), the I-committee (or I-group), the P-committee, and the M-committee. The word MISSISSIPPI has 11 letters, so we then have

$$\frac{11!}{4!4!2!1!} = 34,650$$

Technology Option

```
11!/(4!*4!*2!)
34650
20!/(5!*10!*3!*2!)
465585120
```

EXAMPLE 7 Arrangement of Books A shelf in the book room of a math department has 20 books for its teachers. There are 5 of the same algebra books, 10 of the same finite math books, 3 of the same geometry books, and 2 of the same calculus books. How many distinguishable ways can these 20 books be arranged on the shelf?

Solution From the set of 20 elements we have

$$\frac{20!}{5!10!3!2!} = 465,585,120$$

Optional: A Nonintuitive Example

We now calculate the probability of a certain event and obtain a very surprising (nonintuitive) result.

EXAMPLE 8 The Birthday Problem Suppose there are 50 people in a room with you. What is the probability that at least 2 of these people will have the same birthday?

Solution Suppose more generally there are n people in the room. Let E be the event that at least 2 of these people will have the same birthday. We will ignore leap years and assume that every one of the 365 days of the year is just as likely to be a birthday as any other. It is much easier to first find $P(E^c)$, the probability that no two have the same birthday.

First notice that there are 365 possible birthdays for each individual. Thus, by the multiplication principle, there are 365^n possible birthdays for the n individuals. This is the total number in the sample space. To find $P(E^c)$ notice that there are 365 possible birthdays for the first individual and since the second individual cannot have the same birthday as the first, there are 364 possible birthdays for the second, and then 363 for the third, and so on. We have

n	$P(E)$
15	0.25
20	0.41
23	0.51
30	0.71
50	0.97

$$P(E) = 1 - P(E^c) = 1 - \frac{(365)(364) \cdots (365 - n + 1)}{365^n}$$

The table on the left gives this for several values of n . Notice the very surprising result that for n equal to only 23 the probability is about 0.51. With 50 people in the room the probability is approximately 0.97 that 2 or more of these people will have the same birthday. ♦

Self-Help Exercises 2.3

- A lotto game consists of picking (in any order) the correct 6 numbers drawn from 1 to 42 without replacement.
 - What is the probability of any 1 pick winning?
 - What is the probability of a winning pick being all consecutive numbers?
- A group of five boys and five girls are lining up for lunch. What is the probability that boys and girls alternate?

2.3 Exercises

In Exercises 1 and 2, let an urn have 10 balls, identical except that 4 are white and 6 are red.

- If 3 are selected randomly without replacement, what is the probability that 2 are white and 1 is red? At least 2 are white?
- If 5 are selected randomly without replacement, what is the probability that 3 are white and 2 are red? At least 3 are white?

In Exercises 3 through 6, let an urn have 21 identical balls except that 6 are white, 7 are red, and 8 are blue.

- What is the probability that there is one of each color if 3 are selected randomly without replacement?
- What is the probability that all are white if 3 are selected randomly without replacement?
- What is the probability that 3 are white, 2 are red, and 1 is blue if 6 are selected randomly without replacement?

- What is the probability that at least 5 are white if 6 are selected randomly without replacement?

In Exercises 7 through 12 a 2-card hand is drawn from a standard deck of 52 cards. Find the probability that the hand contains the given cards.

- two kings
- two spades
- a pair
- two of the same suit
- two consecutive cards
- no face card

In Exercises 13 through 20, find the probability of obtaining each of the given in a 5-card poker hand. **Hint:** The probabilities increase.

- royal flush: ace, king, queen, jack, ten in the same suit

14. straight flush: five cards in sequence in the same suit but not a royal flush
15. four of a kind: four queens, four sevens, etc.
16. full house: three of a kind together with a pair
17. straight: five cards in sequence not all in the same suit
18. three of a kind
19. two pairs
20. one pair
21. Assume that the probability of an individual being born in any month is the same and that there are n individuals in a room. Find the probability that at least two individuals have their birthdays in the same month when $n = 2, 3, 4, 5$.
22. Suppose n different letters have been written with n corresponding addressed envelopes, and the letters are inserted *randomly* into the envelope. What is the probability that no letter gets into its correct envelope for $n = 2, 3, 4, 5$?
23. What is the probability that at least two members of the 434-member United States House of Representatives have their birthdays on the same day?
24. What is the probability that at least 2 of the 100 Senators of the U.S. Congress have the same birthday?
29. **Testing** A company places a dozen of the same product in one box. Before sealing, three of the products are tested. If any of the three is defective, the entire box will be rejected. Suppose a box has two defective products. What is the probability the box will be rejected?
30. **Awarding of Contracts** Suppose that there are three corporations competing for four different government contracts. If the contracts are awarded randomly, what is the probability that each corporation will get a contract?
31. A manufacturing company buys a certain component from three different vendors. In how many ways can the company order eight components with four from the first vendor and two each from the other vendors?
32. A mutual fund has 20 stocks in its portfolio. On a given day 3 stocks move up, 15 stay the same, and 2 move down. In how many ways could this happen?
33. An advertising firm has 12 potential clients and three different salesmen. In how many ways can it divide the potential clients equally among the three salesmen?
34. In how many ways can a class of 10 students be assigned 1 A, 2 B's, 4 C's, 2 D's, and 1 F?

Applications

25. **Quality Control** A bin has four defective transistors and six non-defective ones. If two are picked randomly from the bin, what is the probability that both are defective?
26. **Stock Selection** Among a group of 20 stocks, suppose that 10 stocks will perform above average and the other 10 below average. If you pick 3 stocks from this group, what is the probability that all 3 will be above average in performance?
27. **Mutual Funds** Suppose in any year a certain mutual fund is just as likely to perform above average as not. Find the probability that this fund will perform above average in at least 8 of the next 10 years.
28. **Committees** A committee of three is to be selected at random from a group of three senior and four junior executives. What is the probability that the committee will have more senior than junior executives?
35. Two scholarships of \$10,000 each, three of \$5000 each, and five of \$2000 each are to be awarded to 10 finalists. In how many ways can this be done?
36. The 12 directors of a company are to be divided equally into three separate committees to study sales, recent products, and labor relations. In how many ways can this be done?
37. Find the number of arrangements of each of the following words that can be distinguished.
 - a. a_1a_2b
 - b. aab
 - c. $a_1a_2b_1b_2$
 - d. aab_1b_2
 - e. $aabb$
38. Find the number of arrangements of each of the following words that can be distinguished.
 - a. $a_1a_2b_1b_2b_3$
 - b. $aab_1b_2b_3$
 - c. $aabbb$
39. Find the number of arrangements of the word TENNESSEE that can be distinguished.

40. Suppose a word has n symbols made from k distinct elements, with n_1 of the first element, n_2 of the second element, \dots , n_k of the k th element. If $n_1 + n_2 + \dots + n_k = n$, show that the number of distin-

guishable arrangements of the n symbol word is

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

Verify that this works for the previous exercise.

Solutions to Self-Help Exercises 2.3

1. Since the numbers can be drawn in any order, we will use combinations to find the probability.

- a. The number in the sample space S is the number of ways of selecting 6 objects (without replacement) from a set of 42 where order is not important. This is

$$C(42, 6) = \frac{42 \cdot 41 \cdot 40 \cdot 39 \cdot 38 \cdot 37}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} = 5,245,786$$

Thus the probability of any one pick is $\frac{1}{5,245,786}$.

- b. The picks in which the numbers are consecutive are

$$\{1, 2, 3, 4, 5, 6\}, \dots, \{37, 38, 39, 40, 41, 42\}$$

There are 37 such selections. Thus the probability of any one of these being the winning number is $\frac{37}{5,245,786}$.

2. The number in the sample space is the number of ways that these 10 children can line up without any restrictions, so $n(S) = 10! = 3,628,800$. $n(E)$ will be the number of ways the children can line up with boys and girls alternating. This can be found using the multiplication principle and realizing that a boy or girl can be first in line so that there are 10 ways to complete the task of choosing who is first in line. The next person has to be a boy, if a girl was first or a girl, if a boy was first, so there will be 5 ways to complete the task of choosing who is second in line. This gives us $10 \cdot 5 \cdot 4 \cdot 4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = 28,800$. Therefore the probability that boys and girls alternate is

$$P(E) = \frac{n(E)}{n(S)} = \frac{28,800}{3,628,800} = \frac{1}{26} \approx 0.008$$

2.4 Bernoulli Trials

APPLICATION Finding Probabilities of Non-defective Microchips

A computer manufacturer uses eight microchips in each of its computers. It knows that 5% of these chips are defective. What is the probability that at least six are good? See Example 3 for the answer.

◇ Bernoulli Trials

In this section we consider the simplest possible experiments: those with just two outcomes. We refer to experiments in which there are just two outcomes as **Bernoulli trials**. Some examples are as follows.

- Flip a coin and see if heads or tails turns up.
- Test a transistor to see if it is defective or not.
- Examine a patient to see if a particular disease is present or not.
- Take a free throw in basketball and make the basket or not.

We commonly refer to the two outcomes of a Bernoulli trial as “success” (S) or “failure” (F). We agree always to write p for the probability of “success” and q for the probability of “failure.” Naturally, $q = 1 - p$.

In this section we are actually not so much interested in performing an experiment with two outcomes once, but rather many times. We refer to this as a **repeated Bernoulli trial**. We make the following very fundamental assumption.

Fundamental Assumption for Bernoulli Trials

Successive Bernoulli trials are independent of one another.

Thus, for example, flipping a coin 10 times is a repeated Bernoulli trial. Tossing a die 20 times and seeing if an even number or an odd number occurs each time is another example. Consider randomly selecting a card from a standard deck and noting if it is an ace. If we repeat this experiment, but always first replace any card drawn, then the trials are independent and the probability of selecting an ace stays the same. But if we do **not** replace any card drawn, then the probability of selecting an ace changes, and the trials are not independent of each other.

Given a Bernoulli trial repeated n times, we are interested in determining the probability that a specific number of successes occurs. We will often shorten this to $P(k = x)$.

EXAMPLE 1 Making Free Throws Suppose a basketball player makes on average 2 free throws of every 3 attempted and that success and failure on any 1 free throw does not depend on the outcomes of the other shots. If the player shoots 10 free throws, find the probability of making exactly 6 of them.

Solution We let S designate “making the basket” and F “not making the basket.” A typical sequence of exactly six successes looks like

$SSFSFSSFFS$

Now the product rule indicates that the probability of this occurring is

$$\left(\frac{2}{3}\right) \left(\frac{2}{3}\right) \left(\frac{1}{3}\right) \left(\frac{2}{3}\right) \left(\frac{1}{3}\right) \left(\frac{2}{3}\right) \left(\frac{2}{3}\right) \left(\frac{1}{3}\right) \left(\frac{1}{3}\right) \left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^4$$

We might think of the process of obtaining exactly six successes as lining up 10 boxes in a row and picking exactly 6 of them in which to place an S . Thus the previous sequence of successes and failures would be

S S S S S S

Since every such sequence must contain exactly six S 's and four F 's, the probability of any one of these occurring is always $(2/3)^6(1/3)^4$. We then must count the number of ways there can be exactly six successes. But this is the number of ways of selecting six of the boxes to place an S inside. This can be done in $C(10,6)$ ways. Thus

$$\begin{aligned} C(10,6) \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^4 &= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^4 \\ &= 210 \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^4 \\ &\approx 0.228 \end{aligned}$$

In general, to find the probability of exactly k successes in n repeated Bernoulli trials, first notice that any particular sequence with exactly k successes have exactly, $n - k$ failures. Thus by the product rule, the probability of any one of these occurring is

$$p^k q^{n-k}$$

The number of ways of obtaining exactly k successes in n trials is the number of ways of selecting k objects (the boxes with S inside a box in the above discussion) from a total of n . This is $C(n,k)$. The probability of obtaining exactly k successes in n trials is then

$$C(n,k) p^k q^{n-k}$$

REMARK: Be careful to clearly designate which of the outcomes is success and which is failure. Confusing the two will result in using the above formula incorrectly.

◆ Applications

EXAMPLE 2 Baseball Hits What was the probability of a baseball player who has a 0.300 batting average getting at least two hits in a game if we assume that he came to bat four official times in that game and if we assume coming to bat each time is an independent trial.

Solution If we designate success as a hit, then $p = 0.300$ and $q = 1 - p = 0.700$. The probability $P(k)$, where k is the number of successes, is then

$$P(k) = C(n,k) p^k q^{n-k} = C(4,k) (0.300)^k (0.700)^{4-k}$$

We then are looking for $P(2) + P(3) + P(4)$. We have

$$P(2) = C(4,2) (0.3)^2 (0.7)^2 = 6(0.09)(0.49) = 0.2646$$

$$P(3) = C(4,3) (0.3)^3 (0.7)^1 = 4(0.027)(0.7) = 0.0756$$

$$P(4) = C(4,4) (0.3)^4 (0.7)^0 = 1(0.0081)(1) = 0.0081$$

Technology Option

See Technology Note 1 on page 109 for details on finding probabilities in repeated Bernoulli trials using a graphing calculator or a spreadsheet.

Quick Fact: the probability can be found on the TI graphing calculator by going to DISTR which is above the **VAR**s button and scrolling down the **A:binompdf**(option. Select this and then enter the number number of trials, n , a comma then p followed by another comma and finally k , the number of successes. Close the parentheses and press **ENTER**.

```
binompdf(10,2/3,
6)
.2276075801
```

Then

$$P(2) + P(3) + P(4) = 0.2646 + 0.0756 + 0.0081 = 0.3483$$

So about 35% of the time the player should have obtained at least two hits out of four times-at-bat. ♦

EXAMPLE 3 Defective Microchips A computer manufacturer uses eight microchips in each of its computers. It knows that 5% of these chips are defective. What is the probability that

- a. all eight chips are good?
- b. the first three chips are good and one of the last five is defective?

Solution If we let S be the event “not defective,” then $p = 0.95$ and $q = 0.05$.

- a. We have exactly eight successes, so $k = 8$ and the answer is

$$C(8, 8)(0.95)^8(0.05)^0 = (0.95)^8 \approx 0.66342$$

- b. There are two events here. The first event is to find the probability that the first three chips are all good and the second event is to find the probability that exactly one of the last five is defective. These probabilities are independent of each other, so we can multiply them to find the probability that they both occur. That is

$$P = C(3, 3)(0.95)^3(0.05)^0 \times C(5, 4)(0.95)^4(0.05)^1 \\ = 0.85738 \times 0.20363 \approx 0.17458$$

```

Technology Option
binomPdf(8, .95, 8
)
.6634204313
binomPdf(3, .95, 3
)*binomPdf(5, .95
, 4)
.174584324
    
```

◆ **Technology Corner**

🔗 **Technology Note 1 Binomials**

The TI-83/84 calculators have built-in functions for many probability distributions, including the binomial distribution. To access the distribution functions, press the **2ND** and **VAR** buttons to access the DISTR menu. Scroll down until the `A:binomialpdf(` is shown. See Screen 1.

```

DRAW
7: X^Y Pdf(
8: X^Y cdf(
9: Fpdf(
0: Fcdf(
binomialpdf(
B: binomcdf(
C: poissonpdf(
    
```

Screen 1

```

binomialpdf(4, .25)
(.31640625 .421
binomialpdf(4, .25, 0
)
.31640625
    
```

Screen 2

```

binomialcdf(4, .25, 1
)
.73828125
binomialpdf(4, .25, 0
)+binomialpdf(4, .25
, 1)
.73828125
    
```

Screen 3

The `binomialpdf` command has two required values and one optional value. The first value is the number of trials and the second value is the probability of success in a single trial. If the optional third value with the number of successes is not entered, all the values are calculated. See Screen 2 for an experiment with 4 trials and probability of success $p = 0.25$ in each trial. The first value shown in the row under the command is the probability of 0 successes. The left and right arrows will let you view all the results. If a third value, the number of success, is entered in the `binomialpdf` command, then only that probability is calculated. This is also shown on Screen 2.

To find the cumulative probability, use the `binomcdf` command. This function finds the probability of at most k successes. The number of trials is entered first followed by the probability of success in a single trial and then the maximum number of successes. Screen 3 shows the probability of at most one success in a binomial experiment with 4 trials and probability of success 0.25 in a single trial. This value is verified in Screen 3 by explicitly adding the probability of 0 and 1 successes.

A	
1	=BINOMDIST(0,4,0.25,FALSE)
2	=BINOMDIST(1,4,0.25,TRUE)

A	
1	0.31640625
2	0.73828125

A spreadsheet such as Excel can also calculate binomial probability. In Excel the command is `BINOMDIST`. This function requires the number of successes first, followed by the number of trials, the probability of success in a single trial and finally if the cumulative probability is to be calculated or not. This is shown in Worksheet 1.

Worksheet 1

Self-Help Exercises 2.4

1. If a repeated Bernoulli trial is performed six times, find the probability of obtaining two successes and four failures if $p = 0.20$.
2. A retail store sells two brands of TVs, with the first brand comprising 60% of these sales. What is the probability that the next five sales of TVs will consist of at most one of the first brand?

2.4 Exercises

For Exercises 1 through 6, a repeated Bernoulli trial is performed. Find the probability of obtaining the indicated number of successes and failures for the indicated value of p .

1. 4 S 's, 1 F , $p = 0.2$
2. 4 S 's, 3 F 's, $p = 0.3$
3. 3 S 's, 4 F 's, $p = 0.5$
4. 4 S 's, 4 F 's, $p = 0.5$
5. 4 S 's, 4 F 's, $p = 0.25$
6. 2 S 's, 2 F 's, $p = 0.1$

In Exercises 7 through 12, find the probability of exactly k successes in n repeated Bernoulli trials where the probability of success is p .

7. $n = 6$, $k = 3$, $p = 0.5$
8. $n = 6$, $k = 4$, $p = 0.5$
9. $n = 7$, $k = 4$, $p = 0.1$
10. $n = 4$, $k = 3$, $p = 0.2$

11. $n = 5$, $k = 3$, $p = 0.1$

12. $n = 8$, $k = 3$, $p = 0.2$

In Exercises 13 through 18, flip a fair coin 10 times. Find the probability of getting the following outcomes.

13. exactly eight heads
14. exactly three heads
15. at least eight heads
16. at least seven heads
17. at most one head
18. at most two heads

In Exercises 19 through 23, an event E has probability $p = p(E) = 0.6$ in some sample space. Suppose the experiment that yields this sample space is repeated seven times and the outcomes are independent. Find the probability of getting the following outcomes.

19. E exactly six times
20. E exactly three times

21. E at least six times
22. E at least five times
23. E at most two times
24. Show that the probability of exactly $n - k$ successes in n repeated Bernoulli trials where the probability of success is $1 - p$ is the same as the probability of exactly n successes in n repeated Bernoulli trials where the probability of success is p . **Hint:** Use the formula $C(n, k) = C(n, n - k)$.

Ty Cobb has the highest lifetime batting average of any big league baseball player with a remarkable average of .367. Assume in Exercises 25 through 28 that Cobb came to bat officially four times in every game played.

25. What would be Cobb's probability of getting at least one hit in a game?
26. What would be Cobb's probability of getting at least three hits in a game?
27. What would be Cobb's probability of getting at least one hit in 10 successive games? (Use the result in Exercise 25.)
28. What would be Cobb's probability of getting at least one hit in 20 successive games? (Use the result in Exercise 25.)

Babe Ruth holds the record for the highest lifetime percent (8.5%) of home runs per times-at-bat. Assume in Exercises 29 through 32 that Ruth came to bat officially four times in every game played.

29. What would be Ruth's probability of getting at least two home runs in a game?
30. What would be Ruth's probability of getting at least one home run in a game?
31. What would be Ruth's probability of getting at least two home runs in three successive games? (Use the result in Exercise 29.)
32. What would be Ruth's probability of getting four home runs in a game?

Applications

Oil Drilling. An oil company estimates that only 1 oil well in 20 will yield commercial quantities of oil. Assume that successful drilled wells represent independent event. If 12 wells are drilled, find the probability of obtaining a commercially successful well for the following number of times.

33. exactly 1
34. none
35. at most 2
36. exactly 4

Personnel. A company finds that one out of five workers it hires turns out to be unsatisfactory. Assume that the satisfactory performance of any hired worker is independent of that of any other hired workers. If the company hires 20 people, what is the probability that the following number of people will turn out satisfactory?

37. exactly 10
38. at most 2
39. at least 18
40. exactly 20

Medicine. A certain type of heart surgery in a certain hospital results in mortality in 5% of the cases. Assume that the death of a person undergoing this surgery is independent from the death of any others who have undergone this same surgery. If 20 people have this heart surgery at this hospital, find the probability that the following number of people will not survive the operation.

41. exactly 2
42. at most 2
43. at most 3
44. exactly 10

Solutions to Self-Help Exercises 2.4

1. In a repeated Bernoulli trial the probability of obtaining 2 successes and 4 failures if $p = 0.20$ is

$$\begin{aligned} C(n, k)p^kq^{n-k} &= C(6, 2)(0.20)^2(0.80)^4 \\ &= \frac{6 \cdot 5}{2}(0.20)^2(0.80)^4 \\ &\approx 0.246 \end{aligned}$$

Technology Option

```
binompdf(6, .2, 2)
.24576
binomcdf(5, .6, 1)
.08704
```

2. If a retail store sells two brands of TVs, with the first brand comprising 60% of these sales, then the probability that the next five sales of TVs will consist of at most one of the first brand is the probability that the next five sales will consist of exactly zero of the first brand plus exactly one. This is

$$\begin{aligned} C(5, 0)(0.60)^0(0.40)^5 + C(5, 1)(0.60)^1(0.40)^4 \\ &= (0.40)^5 + 5(0.60)(0.40)^4 \\ &= 0.01024 + 0.0768 + 0.08704 \\ &= 0.08704 \end{aligned}$$

2.5 Binomial Theorem

APPLICATION

Number of Ways of Dividing a Committee

A study committee reports to an executive that to solve a particular problem any number of four different actions can be taken, including doing nothing. How many options does the executive have? See Example 3 for the answer.

◇ The Binomial Theorem

The expansion of $(x + y)^2$ is familiar,

$$(x + y)^2 = x^2 + 2xy + y^2$$

In this section formulas for the expansion of $(x + y)^n$ where n is any integer will be given.

We wish to develop a systemic way of writing the expansion of expressions of the form $(x + y)^n$ where n is a positive integer.

First calculating by direct multiplication we can obtain

$$\begin{aligned} (x + y)^0 &= 1 \\ (x + y)^1 &= x + y \\ (x + y)^2 &= x^2 + 2xy + y^2 \\ (x + y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\ (x + y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \\ (x + y)^5 &= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5 \\ (x + y)^6 &= x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6 \end{aligned}$$

Notice that in the expression $(x + y)^n$ the powers of x decrease by one and the powers of y increase by one as we move to the next term. Also notice that for the expression $x^a y^b$, we have $a + b = n$.

How can we predict the coefficients of such terms? Take $(x+y)^5$ as an example. Write

$$(x+y)^5 = (x+y)(x+y)(x+y)(x+y)(x+y) \quad (1)$$

as the product of five factors. We can obtain an x^2y^3 term in the product by selecting y from exactly three of the factors on the right of (1). We can think of this as filling in the blanks of

$$\{_, _, _, _, _\}$$

with exactly three y 's. (The other blanks then must be x .) For example, $\{y, x, y, y, x\}$ indicates that we have selected y from only the first, third, and fourth factors on the right of (1). The number of ways we can select exactly three blanks to put a y in from the five possible blanks is $C(5, 3)$. This is

$$C(5, 3) = \frac{5 \cdot 4 \cdot 3}{3 \cdot 2} = 10$$

which agrees with the coefficient of x^2y^3 in the above expression of $(x+y)^5$.

In general, when looking at the expansion

$$(x+y)^n = (x+y)(x+y)\cdots(x+y)$$

we will obtain an $x^{n-k}y^k$ term by selecting y from exactly k of the n factors. This can be done in $C(n, k)$ ways. Thus the coefficient of the $x^{n-k}y^k$ term must be $C(n, k)$. We have proven the following theorem.

The Binomial Theorem

The coefficient of $x^{n-k}y^k$ in the expansion of $(x+y)^n$ is $C(n, k)$.

EXAMPLE 1 Using the Binomial Theorem Find the coefficient of x^7y^3 in the expansion $(x+y)^{10}$.

Solution According to the binomial theorem this must be

$$C(10, 3) = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2} = 120 \quad \blacklozenge$$

It is common to use the notation

$$\binom{n}{k} = C(n, k)$$

With this notation we then have

$$\begin{aligned} (x+y)^n &= \binom{n}{n} x^n + \binom{n}{n-1} x^{n-1}y + \binom{n}{n-2} x^{n-2}y^2 + \dots \\ &\quad + \binom{n}{2} x^2y^{n-2} + \binom{n}{1} xy^{n-1} + \binom{n}{0} y^n \end{aligned}$$

As an exercise you can show that the coefficient of x^n and y^n are both 1 and that the coefficient of $x^{n-1}y$ and xy^{n-1} are both n .

EXAMPLE 2 Using the Binomial Theorem Write out the expansion of $(a - 2b)^4$.

Solution By setting $x = a$ and $y = -2b$ in the binomial theorem and using the above notation, we have

$$\begin{aligned}(a - 2b)^4 &= (a + (-2b))^4 \\ &= \binom{4}{0} a^4 + \binom{4}{1} a^3(-2b) + \binom{4}{2} a^2(-2b)^2 \\ &\quad + \binom{4}{3} a(-2b)^3 + \binom{4}{4} (-2b)^4 \\ &= a^4 + 4a^3(-2b) + 6a^2(-2b)^2 + 4a(-2b)^3 + (-2b)^4 \\ &= a^4 - 8a^3b + 24a^2b^2 - 32ab^3 + 16b^4\end{aligned}$$

A consequence of the binomial theorem is the following.

The Number of Subsets of a Set

A set with n distinct elements has 2^n distinct subsets.

Before giving a proof, let us list all the subsets of $\{a, b, c\}$ by listing all subsets with three elements, all with two elements, all with one element, and all with no elements. We have 8, which is $8 = 2^3$.

$$\begin{aligned}&\{a, b, c\} \\ &\{a, b\}, \{a, c\}, \{b, c\} \\ &\{a\}, \{b\}, \{c\} \\ &\emptyset\end{aligned}$$

To establish the theorem, the total number of subsets of a set with n distinct elements is the number of subsets with n elements, plus the number of subsets with $(n - 1)$ elements, plus the number with $(n - 2)$ elements, and so on. This is just

$$\binom{n}{n} + \binom{n}{n-1} + \binom{n}{n-2} + \cdots + \binom{n}{1} + \binom{n}{0}$$

Now setting $x = y = 1$ in the binomial theorem gives

$$\begin{aligned}2^n &= (1 + 1)^n \\ &= \binom{n}{n} (1)^n + \binom{n}{n-1} (1)^{n-1} (1) + \binom{n}{n-2} (1)^{n-2} (1)^2 + \cdots \\ &\quad + \binom{n}{2} (1)^2 (1)^{n-2} + \binom{n}{1} (1) (1)^{n-1} + \binom{n}{0} (1)^n \\ &= \binom{n}{n} + \binom{n}{n-1} + \binom{n}{n-2} + \cdots + \binom{n}{1} + \binom{n}{0}\end{aligned}$$

which, as we have just seen, is the total number of subsets we are seeking.

EXAMPLE 3 The Number of Ways of Dividing a Committee A study committee reports to an executive that to solve a particular problem any number of four different actions can be taken, including doing nothing. How many options does the executive have?

Solution The executive has the option of selecting any subset from a set with 4 elements in it. This can be done in $2^4 = 16$ ways. ♦

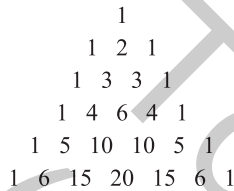


Figure 2.49

◆ Pascal's Triangle

Figure 2.49 lists the coefficients from the expression of $(x + y)^n$ given at the beginning of this section. This is called **Pascal's Triangle**, named after its discoverer, Blaise Pascal (1623–1662). Notice that there are always 1's at the two sides and that any coefficient inside the triangle can be obtained by adding the coefficient above and to the left with the coefficient above and to the right. This is another way the coefficients can be obtained.

Self-Help Exercise 2.5

- Find the coefficient of x^4y^5 in the expansion of $(x + y)^9$.

2.5 Exercises

In Exercises 1 through 10, expand using the binomial theorem.

- | | |
|---------------------|--------------------|
| 1. $(a - b)^5$ | 2. $(2a + b)^4$ |
| 3. $(2x + 3y)^5$ | 4. $(3x - 2y)^4$ |
| 5. $(1 - x)^5$ | 6. $(2 + x)^6$ |
| 7. $(2 - x^2)^4$ | 8. $(1 + 2x)^6$ |
| 9. $(s^2 + t^2)^6$ | 10. $(s^2 - 1)^5$ |
| 11. $(x^2 + y^3)^5$ | 12. $(2x - y^2)^5$ |

In Exercises 13 through 22, determine the first three and last three terms in the expansion of each of the expressions.

- | | |
|----------------------|----------------------|
| 13. $(a - b)^{10}$ | 14. $(a + b)^{12}$ |
| 15. $(x + y)^{11}$ | 16. $(x - y)^8$ |
| 17. $(1 - z)^{12}$ | 18. $(1 + x)^{10}$ |
| 19. $(1 - x^3)^{12}$ | 20. $(x^2 - 1)^{10}$ |
| 21. $(2a + b)^{10}$ | 22. $(a - 2b)^{12}$ |
23. Determine the next row (row 7) in Pascal's Triangle in Figure 2.49.

- Find the coefficient of x^4y^4 in the expansion of $(x + y)^8$ using Pascal's Triangle.

Applications

- A restaurant offers a sundae to which any number of four possible toppings can be added. How many different sundaes can be ordered?
- A restaurant offers a pizza to which any number of six possible toppings can be added. How many different pizzas can be ordered?
- A class ring can be ordered in white or yellow gold, in men's or women's style, with or without a diamond and with or without engraving on the inside and a dark or light finish. How many different class rings are possible?
- A letter jacket has the following options that can be included or not with the jacket: name, class year, academic patch, fine arts patch, athletics patch, student council patch, and FFA patch. How many different letter jackets are possible?

Solution to Self-Help Exercise 2.5

1. Using the binomial theorem, we find the coefficient of x^4y^5 in the expansion of $(x+y)^9$ is

$$C(9,5) = \frac{9!}{5!(9-5)!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{5 \cdot 4 \cdot 3 \cdot 2} = 126$$

Review

◇ Summary Outline

- **General Multiplication Principle.** Suppose there is an operation that consists of making a sequence of k choices with n_1 possible outcomes for the first choice, and, no matter what the first choice, there are always n_2 possible outcomes for the second choice, and, no matter what the first two choices, there are always n_3 possible outcomes for the third choice, and so on. Then there are $n_1 \cdot n_2 \cdots n_k$ possible ways in which the sequence of k choices can be made.

- For any natural number n

$$n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$$

$$0! = 1$$

- A **permutation** of a set of elements is an ordered arrangement of all the elements.
- A **permutation of r objects taken from a set of size n** is a selection of r of the objects with order being important.
- The number of permutations of n distinct objects is given by

$$P(n,n) = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1 = n!$$

- The number of permutations of r distinct objects taken from a set of size n is given by

$$P(n,r) = n(n-1)(n-2) \cdots (n-r+1) = \frac{n!}{(n-r)!}$$

- A **combination** of r distinct objects taken from a set of size n is a selection of r of the objects (without concern for order).
- The number of combinations of r distinct objects taken from a set of size n , denoted by $C(n,r)$, is given by

$$C(n,r) = \frac{n!}{(r!)(n-r)!}$$

- The **Binomial Theorem**. The coefficient of $x^{n-k}y^k$ in the expansion of $(x+y)^n$ is $C(n, k)$.
- Given a set S of n elements, an ordered partition of S of type (n_1, n_2, \dots, n_k) is a division of S into k subsets, S_1, S_2, \dots, S_k , with order being important and where $n_1 = n(S_1), n_2 = n(S_2), \dots, n_k = n(S_k)$. The number of such ordered partitions is

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

- The probability of exactly k successes in n repeated Bernoulli trials where the probability of success is p and failure is q is given by $C(n, k)p^kq^{n-k}$

Review Exercises

- Find $P(10, 4)$ and $C(10, 4)$.
- If instead of a social security *number*, we had a social security *word* where letters could be repeated, what would be the length of the words needed to have at least 250 million different words?
- Before five labor leaders and six management personnel begin negotiating a new labor contract, they decided to first take a picture with the five labor leaders together on the left. In how many different ways can such a picture be taken?
- In the previous exercise suppose a picture is to be taken of three of the labor leaders and three of the management personnel with the labor leaders grouped together on the right. In how many ways can this be done?
- Refer to Exercise 3. Suppose a committee of three labor leaders and four management personnel is formed to study the issue of pensions. In how many ways can this be done?
- Refer to Exercise 3. In a straw vote on a proposal from the labor leaders, the six management personnel each cast a vote with no abstentions. In how many ways can these six individuals come to a majority decision?
- An investor decides that her investment year is a success if her portfolio of stocks beats the S&P 500. In how many ways can her next 10 years have exactly seven successes?
- Expand $(2-x)^5$ using the binomial theorem.
- Find the last three terms in the expansion of $(1-x)^9$.
- In how many ways can a laboratory divide 12 scientists into four groups of equal size in order to perform four different experiments?
- An urn has 10 white, 5 red, and 15 blue balls. What is the probability that there is one of each color if 3 are selected randomly without replacement? With replacement?
- For the urn in the previous exercise, what is the probability that 3 are white, 4 are red, and 2 are blue, if 9 are selected randomly without replacement? With replacement?
- Assembly Line** A machine on an assembly line is malfunctioning randomly and produces defective parts 30% of the time. What is the probability that this machine will produce exactly three defective parts among the next six?
- Teaching Methods** An instructor finds that only 55% of her college algebra students pass the course. She then tries a new teaching method and finds that 85% of the 20 students in the first class with the new method pass. Assuming that the probability at this school of passing college algebra is 0.55 and that any one student passing is independent of any other student passing, what is the probability that at least 85% of a college algebra class of 20 will pass this course? Is the instructor justified in claiming that the new method is superior to the old method?

p	q	r	$\sim q$	$p \vee \sim q$	$\sim q \wedge r$	$(p \vee \sim q) \vee (\sim q \wedge r)$
T	T	T	F	T	F	T
T	T	F	F	T	F	T
T	F	T	T	T	T	T
T	F	F	T	T	F	T
F	T	T	F	F	F	F
F	T	F	F	F	F	F
F	F	T	T	T	T	T
F	F	F	T	T	F	T

21. a. True b. False c. True d. True e. False

23. a. True b. False c. False d. True

1.1 EXERCISES

1. a. False b. False 3. a. True b. False

5. a. False b. True c. False (d) True

7. a. $\emptyset, \{3\}$ b. $\emptyset, \{3\}, \{4\}, \{3,4\}$



9.

11. a. 2 b. 3 disjoint

13. a. 1, 2, 3 b. 1 not disjoint

15. a. I b. V

17. a. VIII b. IV

19. a. II, V, VI b. III, IV, VII

21. a. VII b. I, II, III, IV, V, VII, VIII

23. a. $\{4, 5, 6\}$ b. $\{1, 2, 3, 4, 5, 6, 7, 8\}$

25. a. $\{1, 2, 3\}$ b. $\{9, 10\}$

27. a. $\{5, 6\}$ b. $\{1, 2, 3, 4, 7, 8, 9, 10\}$

29. a. \emptyset b. \emptyset

31. a. People in your state who do not own an automobile

b. People in your state who own an automobile or house

c. People in your state who own an automobile or not a house

33. a. People in your state who own an automobile but not a house

b. People in your state who do not own an automobile and do not own a house

c. People in your state who do not own an automobile or do not own a house

35. a. People in your state who own an automobile and a house and a piano

b. People in your state who own an automobile or a house or a piano

c. People in your state who own both an automobile and a house or else own a piano

37. a. People in your state who do not own both an automobile and a house but do own a piano

b. People in your state who do not own an automobile, nor a house, nor a piano

c. People in your state who own a piano, but do not own a car or a house

39. a. $N \cap F$ b. $N \cap H^c$ 41. a. $N \cup S$ b. $N^c \cap S^c$

43. a. $(N \cap H) \cup (S \cap H)$ b. $(N \cap F) \cap H^c$

45. a. $(F \cap H) \cap (N \cup S)^c$ b. $F \cap H^c \cap N^c \cap S^c$

47. Both expressions give U

49. Both expressions give $\{1, 2, 3, 4, 5, 6, 7\}$

51. Both expressions give $\{8, 9, 10\}$.

1.2 EXERCISES

1. 135 3. 70 5. 60 7. 150 9. 110

11. 90 13. 15 15. 7 17. 3 19. 56

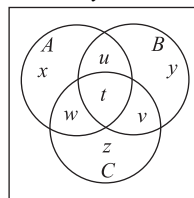
21. a. 1100 b. 750 c. 100

23. a. 100 b. 500

25. a. 30 b. 360 c. 70

27. a. 20 b. 25 c. 345 29. 30

31. From the figure we have $n(A) - n(A \cap B) = x + w$, $n(B) - n(B \cap C) = u + y$, $n(C) - n(A \cap C) = v + z$, and $n(A \cap B \cap C) = t$. Adding these four equations gives the result since from the figure $n(A \cup B \cup C) = t + u + v + w + x + y + z$.



1.3 EXERCISES

1. $\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$

3. $S = \{(H, H, H), (H, H, T), (H, T, H), (T, H, H), (H, T, T), (T, H, T), (T, T, H), (T, T, T)\}$,

$E = \{(H, H, H), (H, H, T), (H, T, H), (T, H, H)\}$

5. $S = \{(1, 1, 1), (1, 1, 0), (1, 0, 1), (0, 1, 1), (0, 0, 1), (0, 1, 0), (1, 0, 0), (0, 0, 0)\}$, $E = \{(1, 1, 1), (1, 1, 0), (1, 0, 1), (0, 1, 1)\}$

7. a. $\{\text{black, white, red}\}$ b. $\{\text{black, red}\}$

9. $\{(A, B, C), (A, B, D), (A, B, E), (A, C, D), (A, C, E), (A, D, E), (B, C, D), (B, C, E), (B, D, E), (C, D, E)\}$, $\{(A, B, C), (A, B, D), (A, B, E), (B, C, D), (B, C, E), (B, D, E)\}$

11. $S = \{1, 2, 3, 4, 6, 9\}$, $E = \{2, 4, 6\}$

13. a. $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ b. $E = \{6, 7, 8, 9, 10\}$ c. $F = \{0, 1, 2, 3, 4\}$

d. $E \cup F = \{0, 1, 2, 3, 4, 6, 7, 8, 9, 10\}$. $E \cap F = \emptyset$. $E^c = \{0, 1, 2, 3, 4, 5\}$. $E \cap F^c = \{6, 7, 8, 9, 10\}$. $E^c \cap F^c = \{5\}$

e. E^c and $E \cap F^c$. $E \cap F^c$ and $E^c \cap F^c$. $E \cup F$ and $E^c \cap F^c$.

15. a. $\{(S, S, S), (S, S, F), (S, F, S), (F, S, S), (S, F, F), (F, S, F), (F, F, S), (F, F, F)\}$

b. $E = \{(S, S, S), (S, S, F), (S, F, S), (F, S, S)\}$

c. $G = \{(S, S, S), (S, S, F), (S, F, S), (S, F, F)\}$

d. $E \cup G = \{(S, S, S), (S, S, F), (S, F, S), (S, F, F), (F, S, S)\}$, $E \cap G = \{(S, S, S), (S, S, F), (S, F, S)\}$,

$G^c = \{(F, S, S), (F, S, F), (F, F, S), (F, F, F)\}$, $E^c \cap G = \{(S, F, F)\}$, $(E \cup G)^c = \{(F, F, S), (F, S, F), (F, F, F)\}$.

e. $E \cap G$ and G^c . $E \cap G$ and $E^c \cap G$. $E \cap G$ and $(E \cup G)^c$. G^c and $E^c \cap G$. $E^c \cap G$ and $(E \cup G)^c$. $E \cup G$ and $(E \cup G)^c$.

17. a. Pencils that are longer than 10 cm and less than 25 cm.

b. Pencils that are less than 10 cm long.

c. Pencils that are 25 cm or longer. d. \emptyset

19. $E \cap F^c$ 21. $F^c \cap E^c$ 23. $E \cap F \cap G^c$

25. $E \cup F \cup G =$ all 26 letters. $E^c \cap F^c \cap G^c = \emptyset$. $E \cap F \cap G = \emptyset$. $E \cup F^c \cup G = \{a, e, i, o, u, b, c, d\}$

27. $S = \{r, b, g, y\}$, $E = \{b, g\}$

1.4 EXERCISES

1. $1/2$ 3. $2/3$ 5. $1/13$ 7. $1/2$
 9. $3/26$ 11. $1/4$ 13. $2/3$ 15. 0.15
 17. a. 0.12 b. 0.8 c. 0.08 19. 0.56
 21. A: 0.125, B: 0.175, C: 0.4, D: 0.2, F: 0.1
 23. 0.55 25. $1/2$ 27. $1/2$ 29. $3/8, 7/8$
 31. $1/12$ 33. $1/36, 1/12, 5/36$ 35. $1/3$

1.5 EXERCISES

1. 0.50, 0.90 3. 0.18, 0.63, 0.62 5. 0.25
 7. $1/6$ 9. $7/30$ 11. 0.60, 0.80, 0
 13. 0.70, 0.70, 0, 0.30, 1 15. 0.60, 0.10
 17. 0.20, 0.10 19. 0.20, 0.10, 0.15
 21. 0.85 23. a. 3:17 b. 1:3
 25. 2:3 27. 7:5 29. 0.20
 31. 0.04, 0.96 33. 0.039, 0.049, 0.951
 35. a. 0.009 b. 0.001 c. 0.006 d. 0.974
 37. 0.20

39. You do not know what the actual probability is. You do know that the empirical probability is $165/1000 = 0.165$. This represents the best guess for the actual probability. But if you tossed the coin more times, the relative frequency and the new empirical probability would most likely had changed.

41. The probabilities in the game are constant and do not change just because you are on a winning streak. Thus no matter what has happened to you in the past, the probability of winning any one game remains constant at 0.48. Thus if you continue to play, you should expect to win 48% of the time in the future. You have been lucky to have won 60% of the time up until now.

43. After reading the first discussion problem above, we know that it is, in fact, impossible to determine with certainty the actual probability precisely. Since the die has been tossed a total of 2000 times and a one has come up 335 times, our best guess at the probability is $335/2000 = 0.1675$.

1.6 EXERCISES

1. $3/7, 3/5$ 3. $2/3, 4/5$ 5. 1, 0
 7. $2/3, 1/2$ 9. $1/3, 1$ 11. 0, $1/2$
 13. No 15. Yes 17. No
 19. Yes 21. $2/11$ 23. $1/7$
 25. a. $\frac{10,200}{132,600} \approx 0.077$ b. $\frac{49}{25.33} \approx 0.059$
 27. 0.12, 0.64, 0.60 29. $1/3, 3/10$
 31. No 33. 0.65 35. 0.72
 37. 0.02, 0.017 39. 0.026
 41. 0.000001 43. $5/7, 5/21, 1/21$
 45. Yes 47. No 49. 0.057818

51. For E and F to be independent, they must satisfy $P(E) \times P(F) = P(E \cap F)$. From the Venn diagram, we must have: $(p_1 + p_2) \times (p_3 + p_2) = p_2$. So,

$$p_1 p_3 + p_2 p_3 + p_1 p_2 + p_2^2 = p_2$$

$$p_1 p_3 = p_2(1 - p_3 - p_2 - p_1) = p_2 p_4$$

The above steps can be reversed, so if $p_1 p_3 = p_2 p_4$, we will have $P(E) \times P(F) = P(E \cap F)$.

If the sets are mutually disjoint, then $p_2 = 0$. This implies that $p_1 p_3 = p_2 p_4 = 0$. Then either p_1 or p_3 or both are zero. Thus either $P(E) = 0$, $P(F) = 0$, or both.

53. Since E and F are independent, $P(E) \times P(F) = P(E \cap F)$.

$$\begin{aligned} P(E^c \cap F^c) &= 1 - P(E \cup F) \\ &= 1 - (P(E) + P(F) - P(E \cap F)) \\ &= 1 - P(E) - P(F) + P(E) \times P(F) \\ &= (1 - P(E)) \times (1 - P(F)) \\ &= P(E^c) \times P(F^c) \end{aligned}$$

Hence, if E and F are independent, so are E^c and F^c .

55. Since E and F are exclusive, $P(E \cap F) = P(\emptyset) = 0$. Since $P(E)$ and $P(F)$ are both nonzero, then $P(E) \times P(F) > 0$. Therefore, E and F are not independent.

57.

$$\begin{aligned} P(E^c|F) &= \frac{P(E^c \cap F)}{P(F)} \\ &= \frac{P(F) - P(E \cap F)}{P(F)} \\ &= 1 - P(E|F) \end{aligned}$$

$$59. P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{P(F)}{P(F)} = 1$$

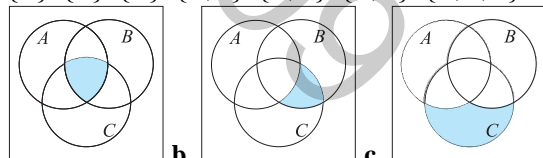
$$61. P(E|F) + P(E^c|F) = \frac{P(E \cap F) + P(E^c \cap F)}{P(F)} = \frac{P(F)}{P(F)} = 1$$

1.7 EXERCISES

1. $3/7, 12/31$ 3. $3/19, 1/3$ 5. $4/11$
 7. a. $2/23$ b. $21/23$ 9. $2/17$
 11. a. $1/56$ b. $5/28$ 13. $2/7$
 15. $4/19$ 17. $13/51$ 19. $13/51$
 21. $6/17$ 23. $100/136 \approx 0.74$
 25. $396/937 \approx 0.42, 5/937 \approx 0.005$
 27. $5/7$ 29. $10/19$ 31. a. $1/12$ b. $1/4$
 33. $P(1|N) = 6/20, P(2|N) = 4/20, P(3|N) = 6/20, P(4|N) = 4/20$

REVIEW EXERCISES

1. a. Yes b. No c. Yes
 2. $\{x \mid x = 5n, n \text{ is an integer and } 1 \leq n \leq 8\}$
 3. $\{0, -\sqrt{2}, \sqrt{2}\}$
 4. $\emptyset, \{A\}, \{B\}, \{C\}, \{A, B\}, \{A, C\}, \{B, C\}, \{A, B, C\}$



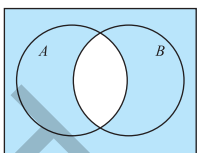
5. a. b. c.
 6. $A \cup B = \{1, 2, 3, 4\}, A \cap B = \{2, 3\}, B^c = \{1, 5, 6\}, A \cap B \cap C = \{2, 3\} \cap \{4, 5\} = \emptyset, (A \cup B) \cap C = \{4\}, A \cap B^c \cap C = \emptyset$
 7. a. My current instructors who are less than 6 feet tall. b. My current instructors who are at least 6 feet tall or are male. c. My current instructors who are female and weigh at most 180 pounds. d. My current male instructors who are at least 6 feet tall and weigh more than 180 pounds. e. My current male instructors who are less than 6 feet tall and weigh more

than 180 pounds. **f.** My current female instructors who are at least 6 feet tall and weigh more than 180 pounds.

8. a. M^c **b.** $M^c \cap W^c$ **c.** $M \cap (H \cup W)$

9. $\{1, 2, 3, 4\} = \{1, 2, 3, 4\}$

10. The shaded area is $(A \cap B)^c = A^c \cup B^c$.



11. 120 **12.** 10 **13.** 50

14. a. 40 **b.** 45 **c.** 19

15. a. 0.016, 0.248, 0.628, 0.088, 0.016, 0.044 **b.** 0.892

c. 0.264

16. a. $5/30$ **b.** $15/30$ **c.** $20/30$

17. $P(E \cup F) = 0.60, P(E \cap F) = 0, P(E^c) = 0.75$

18. $P(E \cup F) = 0.55, P(E^c \cap F) = 0.35,$

$P((E \cup F)^c) = 0.45$

19. $P(B) = 1/6$ **20.** 0.75

21. a. 0.09 **b.** 0.91

22. a. 0.13 **b.** 0.09 **c.** 0.55

23. $P(E|F) = 0.40, P(E^c|F) = 0.60, P(F^c|E^c) = 4/7$

24. Not independent

25. $(0.05)^3$

26. 0.254

27. a. $2/31$ **b.** $4/31$

28. 0.50

2.1 EXERCISES

1. $5 \times 4 \times 3 = 60$ **3.** $8 \times 7 \times 6 \times 5 \times 4 = 6720$

5. $7! = 5040$ **7.** $9! = 362,880$ **9.** $9 \times 8 = 72$

11. n **13.** $4 \times 30 = 120$ **15.** $4 \times 10 \times 5 = 200$

17. $10^6 = 1,000,000$ **19.** $2 \times 10 \times 5 \times 3 = 300$

21. $(26)^3(10)^3 = 17,576,000$ **23.** 96

25. $5 \times 4 \times 4 \times 3 \times 3 \times 2 \times 2 = 2880$ **27.** $5! = 120$

29. $2!4!6! = 34,560$ **31.** $9!5! = 43,545,600$

33. $3!4! = 144$ **35.** $12 \times 11 \times 10 \times 9 = 11,880$

37. 1,814,400 **39.** $(10 \times 9 \times 8)(8 \times 7 \times 6) = 241,920$

41. $P(11, 4) \times P(9, 3) \times P(5, 2) = 79,833,600$

43. $P(n, r) = n(n-1)(n-2) \cdots (n-r+1)$

$$= n(n-1)(n-2) \cdots (n-r+1) \frac{(n-r)(n-r-1) \cdots 2 \cdot 1}{(n-r)(n-r-1) \cdots 2 \cdot 1}$$

$$= \frac{n(n-1)(n-2) \cdots (n-r+1)(n-r)(n-r-1) \cdots 2 \cdot 1}{(n-r)(n-r-1) \cdots 2 \cdot 1}$$

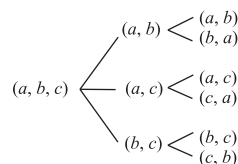
$$= \frac{n!}{(n-r)!}$$

45. 24

2.2 EXERCISES

1. $\frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56$ **3.** $\frac{8 \times 7 \times 6 \times 5 \times 4}{5 \times 4 \times 3 \times 2 \times 1} = 56$ **5.** 12

7. 35 **9.** 105 **11.** n



13.

15. 10

17. $C(46, 6) = \frac{46!}{40! \times 6!} = 9,366,819$

19. $C(20, 5) = \frac{20!}{15! \times 5!} = 15,504$

21. $C(20, 4) = \frac{20!}{16! \times 4!} = 4845$

23. $C(40, 5) = \frac{40!}{35! \times 5!} = 658,008$

25. $C(11, 3) \times C(7, 2) = 3465$

27. $C(20, 7) \times C(6, 3) = 1,550,400$

29. $C(12, 3)P(9, 3) = 110,880$

31. $C(11, 4)P(7, 2) = 13,860$

33. $C(9, 5) = 126$

35. $C(8, 5) = 56$

37. $2^8 - C(8, 0) - C(8, 1) = 256 - 1 - 8 = 247$

39. $C(12, 3)C(8, 3)C(4, 2)2 = 147,840$

41. $C(4, 3) \cdot 48 \cdot 44/2 = 4224$

43. $C(4, 2)C(4, 2)C(13, 2) \cdot 44 = 123,552$

$$\begin{aligned} \mathbf{45.} \quad C(n, r) &= \frac{n!}{(r!)(n-r)!} \\ &= \frac{n!}{(n-r)!(n-(n-r))!} \\ &= C(n, n-r) \end{aligned}$$

2.3 EXERCISES

1. $3/10, 1/3$ **3.** $(6 \cdot 7 \cdot 8)/1330$ **5.** $20/323$ **7.** $6/1326$

9. $78/1326$ **11.** $208/1326$ **13.** $4/2,589,960$

15. $624/2,589,960$ **17.** $10,200/2,589,960$

19. $123,552/2,589,960$

21. 0.0833; 0.2361; 0.4271; 0.6181

23. 1 **25.** $6/45$ **27.** $56/2^{10} \approx 0.0547$

29. $5/11$ **31.** 420 **33.** 34,650 **35.** 2520

37. a. 6 **b.** 3 **c.** 24 **d.** 12 **e.** 6 **39.** 3780

2.4 EXERCISES

1. $5(.2)^4(.8) = .0064$ **3.** $35(.5)^7 \approx 0.273$

5. $70(.25)^4(.75)^4 \approx 0.087$ **7.** $20(.5)^6 \approx 0.3125$

9. $35(.1)^4(.9)^3 \approx 0.00255$ **11.** $10(.1)^3(.9)^2 \approx 0.0081$

13. $45 \times (0.5)^{10} \approx 0.044$ **15.** $56(0.5)^{10} \approx 0.0547$

17. $11(0.5)^{10} \approx 0.0107$ **19.** $7(0.6)^6(0.4) \approx 0.1306$

21. $7(0.6)^6(0.4) + (0.6)^7 \approx 0.1586$

23. $(0.4)^7 + 7(0.6)(0.4)^6 + 21(0.6)^2(0.4)^5 \approx 0.0962$

25. $1 - (0.633)^4 \approx 0.839$ **27.** $(0.839)^{10} \approx 0.173$

29. $1 - (0.915)^4 - 4(0.085)(0.915)^3 \approx 0.03859$

31. $(0.0386)^3 \approx 0.0000575$

33. $12(0.05)(0.95)^{11} \approx 0.341$

35. $(.95)^{12} + 12(.05)(.95)^{11} + 66(.05)^2(.95)^{10} \approx 0.980$

37. $C(20, 10)(0.8)^{10}(0.2)^{10} \approx 0.002$

39. $190(.8)^{18}(.2)^2 + 20(.8)^{19}(.2) + (.8)^{20} \approx 0.206$

41. $190(0.05)^2(0.95)^{18} \approx 0.189$ **43.** ≈ 0.984

2.5 EXERCISES

1. $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$

3. $32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5$

5. $1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6$

7. $16 - 32x^2 + 24x^4 - 8x^6 + x^8$
 9. $s^{12} + 6s^{10}t^2 + 15s^8t^4 + 20s^6t^6 + 15s^4t^8 + 6s^2t^{10} + t^{12}$
 11. $x^8 + 4x^6y^3 + 6x^4y^6 + 4x^2y^9 + y^{12}$
 13. $a^{10} - 10a^9b + 45a^8b^2, 45a^2b^8 - 10ab^9 + b^{10}$
 15. $x^{11} + 11x^{10}y + 55x^9y^2, 55x^2y^9 + 11xy^{10} + y^{11}$
 17. $1 - 12z + 66z^2, 66z^{10} - 12z^{11} + z^{12}$
 19. $1 - 12x^3 + 66x^6, 66x^{30} - 12x^{33} + x^{36}$
 21. $1024a^{10} + 5120a^9b + 11520a^8b^2, 180a^2b^8 + 20ab^9 + b^{10}$
 23. 1, 7, 21, 35, 35, 21, 7, 1 25. 16 27. 32

REVIEW EXERCISES

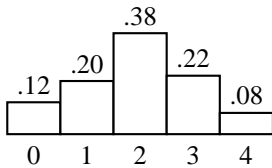
1. 5040, 210 2. 6 3. $5!6! = 86,400$
 4. $P(6,3)P(5,3) = 7200$ 5. $C(5,3) \times C(6,4) = 150$
 6. 44 7. $C(10,7) = 120$
 8. $32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5$ 9. $-36x^7 + 9x^8 - x^9$
 10. 369,600 11. $(10 \cdot 5^{15})/C(30,3) \approx 0.185, 1/6$
 12. $C(10,3) \cdot C(5,4) \cdot C(15,2)/C(30,9) \approx 0.0044,$
 $C(9,3)C(6,4)(10/30)^3(5/30)^4(15/30)^2 \approx 0.009$
 13. $C(6,3)(0.3)^3(0.7)^3 \approx 0.185$ 14. ≈ 0.0049 . Yes.

3.1 EXERCISES

1. infinite discrete 3. continuous
 5. finite discrete 7. infinite discrete

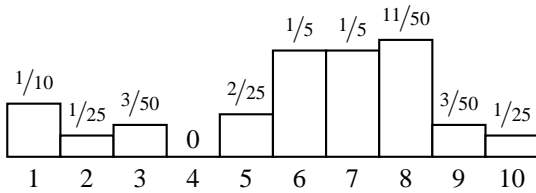
9.

A	B	C	D	F
.12	.20	.38	.22	.08



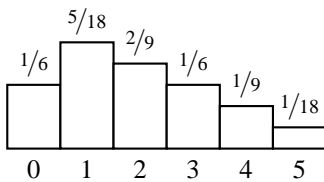
11.

1	2	3	4	5	6	7	8	9	10
.10	.04	.06	0	.08	.20	.20	.22	.06	.04



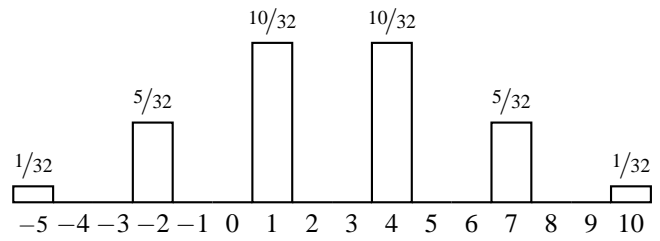
13.

Event	0	1	2	3	4	5
Frequency	6	10	8	6	4	2
$P(X = x)$	$\frac{1}{6}$	$\frac{5}{18}$	$\frac{2}{9}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{18}$



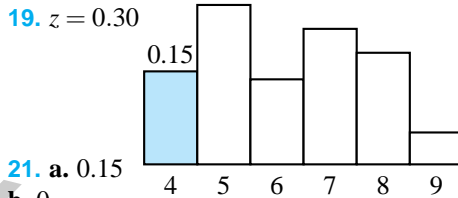
15.

x	-5	-2	1	4	7	10
$P(X = x)$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$

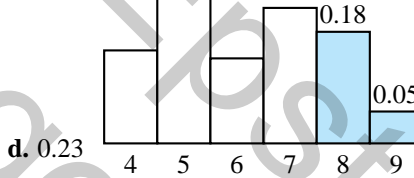
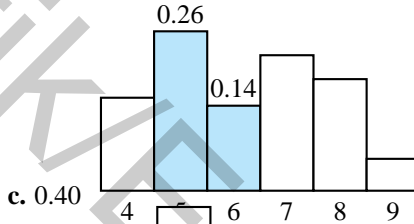


17.

x	0	1	2
$P(X = x)$	$\frac{21}{45}$	$\frac{21}{45}$	$\frac{3}{45}$



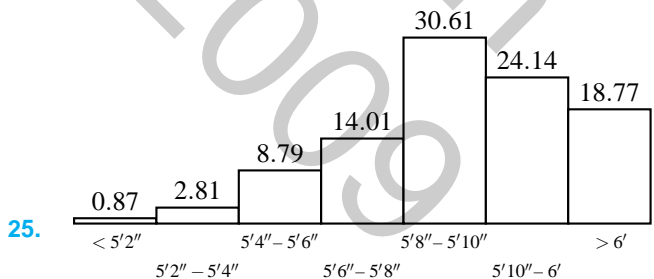
21. a. 0.15
 b. 0



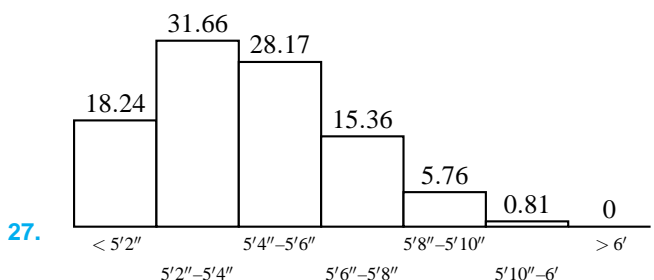
- d. 0.23

23.

y	0	1	2	3	4	5
$P(y = x + 2)$	0.20	0.15	0.05	0.35	0.15	0.10



- 25.



- 27.