LINEAR PROGRAMMING: A GEOMETRIC APPROACH

3

MANY PRACTICAL PROBLEMS involve maximizing or minimizing a function subject to certain constraints. For example, we may wish to maximize a profit function subject to certain limitations on the amount of material and labor available. Maximization or minimization problems that can be formulated in terms of a *linear* objective function and constraints in the form of linear inequalities are called *linear programming problems*. In this chapter, we look at linear programming problems involving two variables. These problems are amenable to geometric analysis, and the method of solution introduced here will shed much light on the basic nature of a linear programming problem.

How many souvenirs should Ace Novelty make in order to maximize its profit? The company produces two types of souvenirs, each of which requires a certain amount of time on each of 2 different machines. Each machine can be operated for only a certain number of hours per day. In Example 1, page 164, we show how this production problem can be formulated as a linear programming problem, and in Example 1, page 175, we solve this linear programming problem.

3.1 Graphing Systems of Linear Inequalities in Two Variables

Graphing Linear Inequalities

In Chapter 1, we saw that a linear equation in two variables x and y

$$ax + by + c = 0 \qquad \textit{a, b not both equal to zero}$$

has a *solution set* that may be exhibited graphically as points on a straight line in the xy-plane. We now show that there is also a simple graphical representation for **linear inequalities** in two variables:

$$\begin{array}{ll} ax + by + c < 0 & \quad ax + by + c \leq 0 \\ ax + by + c > 0 & \quad ax + by + c \geq 0 \end{array}$$

Before turning to a general procedure for graphing such inequalities, let's consider a specific example. Suppose we wish to graph

$$2x + 3y < 6 \tag{1}$$

We first graph the equation $2x + 3y = 6$, which is obtained by replacing the given inequality "<" with an equality "=" (Figure 1).

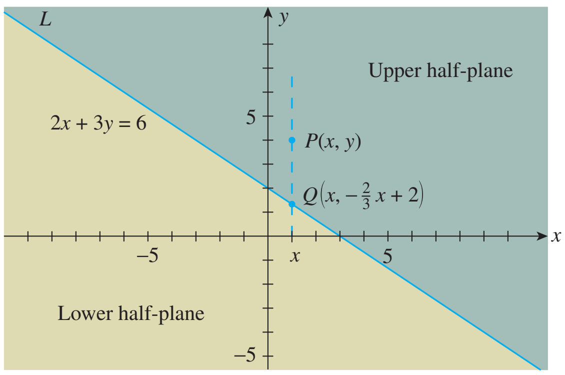


FIGURE 1
A straight line divides the *xy*-plane into two half-planes.

Observe that this line divides the xy-plane into two half-planes: an upper half-plane and a lower half-plane. Let's show that the upper half-plane is the graph of the linear inequality

$$2x + 3y > 6 \tag{2}$$

whereas the lower half-plane is the graph of the linear inequality

$$2x + 3y < 6 \tag{3}$$

To see this, let's write Inequalities (2) and (3) in the equivalent forms

$$y > -\frac{2}{3}x + 2 \tag{4}$$

and

$$y < -\frac{2}{3}x + 2 \tag{5}$$

The equation of the line itself is

$$y = -\frac{2}{3}x + 2 \tag{6}$$

Now pick any point $P(x, y)$ lying above the line L. Let Q be the point lying on L and directly below P (see Figure 1). Since Q lies on L, its coordinates must satisfy Equation (6). In other words, Q has representation $Q(x, -\frac{2}{3}x + 2)$. Comparing the y-coordinates of P and Q and recalling that P lies above Q, so that its y-coordinate must be larger than that of Q, we have

$$y > -\frac{2}{3}x + 2$$

But this inequality is just Inequality (4) or, equivalently, Inequality (2). Similarly, we can show that every point lying below L must satisfy Inequality (5) and therefore (3).

This analysis shows that the lower half-plane provides a solution to our problem (Figure 2). (By convention, we draw the line as a dashed line to show that the points on L do not belong to the solution set.) Observe that the two half-planes in question are disjoint; that is, they do not have any points in common.

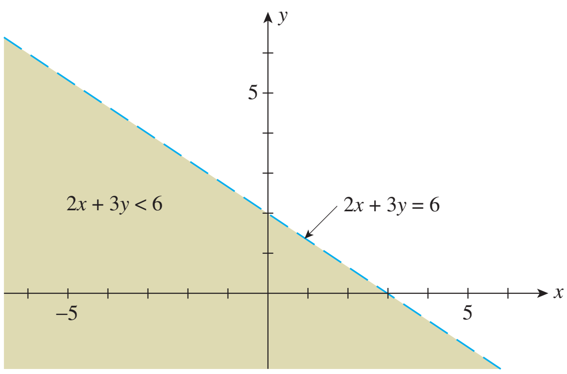


FIGURE 2
The set of points lying below the dashed line satisfies the given inequality.

Alternatively, there is a simpler method for determining the half-plane that provides the solution to the problem. To determine the required half-plane, let's pick *any* point lying in one of the half-planes. For simplicity, pick the origin (0, 0), which lies in the lower half-plane. Substituting $x = 0$ and $y = 0$ (the coordinates of this point) into the given Inequality (1), we find

$$2(0) + 3(0) < 6$$

or $0 < 6$, which is certainly true. This tells us that the required half-plane is the one containing the test point—namely, the lower half-plane.

Next, let's see what happens if we choose the point (2, 3), which lies in the upper half-plane. Substituting $x = 2$ and $y = 3$ into the given inequality, we find

$$2(2) + 3(3) < 6$$

or $13 < 6$, which is false. This tells us that the upper half-plane is *not* the required half-plane, as expected. Note, too, that no point $P(x, y)$ lying on the line constitutes a solution to our problem, given the *strict* inequality $<$.

This discussion suggests the following procedure for graphing a linear inequality in two variables.

Procedure for Graphing Linear Inequalities

1. Draw the graph of the equation obtained for the given inequality by replacing the inequality sign with an equal sign. Use a dashed or dotted line if the problem involves a strict inequality, $<$ or $>$. Otherwise, use a solid line to indicate that the line itself constitutes part of the solution.
2. Pick a test point lying in one of the half-planes determined by the line sketched in step 1 and substitute the values of x and y into the given inequality. For simplicity, use the origin whenever possible.
3. If the inequality is satisfied, the graph of the solution to the inequality is the half-plane containing the test point. Otherwise, the solution is the half-plane not containing the test point.

VIDEO

EXAMPLE 1 Determine the solution set for the inequality $2x + 3y \geq 6$.

Solution Replacing the inequality $\geq$ with an equality $=$, we obtain the equation $2x + 3y = 6$, whose graph is the straight line shown in Figure 3.

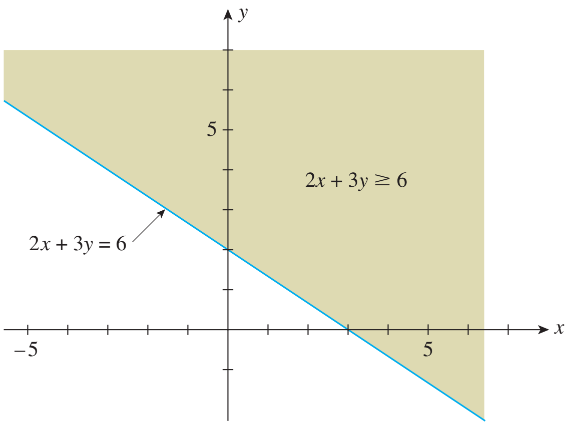


FIGURE 3
The set of points lying on the line and in the upper half-plane satisfies the given inequality.

Instead of a dashed line as before, we use a solid line to show that all points on the line are also solutions to the inequality. Picking the origin as our test point, we find $2(0) + 3(0) \geq 6$, or $0 \geq 6$, which is false. So we conclude that the solution set is made up of the half-plane not containing the origin, including (in this case) the line given by $2x + 3y = 6$.

EXAMPLE 2 Graph $x \leq -1$.

Solution The graph of $x = -1$ is the vertical line shown in Figure 4. Picking the origin $(0, 0)$ as a test point, we find $0 \leq -1$, which is false. Therefore, the required solution is the *left* half-plane, which does not contain the origin.

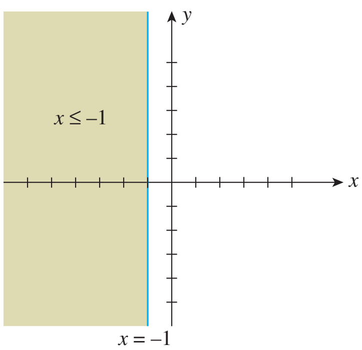


FIGURE 4
The set of points lying on the line $x = -1$ and in the left half-plane satisfies the given inequality.

EXAMPLE 3 Graph $x - 2y > 0$.

Solution We first graph the equation $x - 2y = 0$, or $y = \frac{1}{2}x$ (Figure 5). Since the origin lies on the line, we may not use it as a test point. (Why?) Let's pick $(1, 2)$ as a test point. Substituting $x = 1$ and $y = 2$ into the given inequality, we find $1 - 2(2) > 0$, or $-3 > 0$, which is false. Therefore, the required solution is the half-plane that does not contain the test point—namely, the lower half-plane.

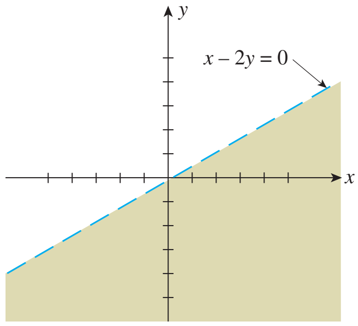


FIGURE 5
The set of points in the lower half-plane satisfies $x - 2y > 0$.

Exploring with TECHNOLOGY

A graphing utility can be used to plot the graph of a linear inequality. For example, to plot the solution set for Example 1, first rewrite the equation $2x + 3y = 6$ in the form $y = 2 - \frac{2}{3}x$. Next, enter this expression for Y_1 in the calculator and move the cursor to the left of Y_1. Then, press **ENTER** repeatedly and select the icon that indicates the shading option desired (see Figure a). The required graph follows (see Figure b).

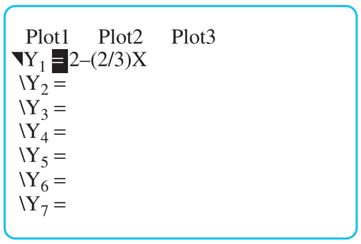


FIGURE a
TI 83/84 screen

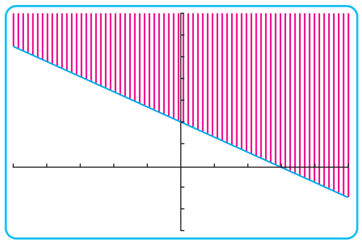


FIGURE b
Graph of the inequality $2x + 3y \geq 6$

Graphing Systems of Linear Inequalities

By the **solution set of a system of linear inequalities** in the two variables x and y we mean the set of all points (x, y) satisfying each inequality of the system. The graphical solution of such a system may be obtained by graphing the solution set for each inequality independently and then determining the region in common with each solution set.

EXAMPLE 4 Determine the solution set for the system

$$\begin{aligned} 4x + 3y &\geq 12 \\ x - y &\leq 0 \end{aligned}$$

Solution Proceeding as in the previous examples, you should have no difficulty locating the half-planes determined by each of the linear inequalities that make up the system. These half-planes are shown in Figure 6. The intersection of the two half-planes is the shaded region. A point in this region is an element of the solution set for the given system. The point P, the intersection of the two straight lines determined by the equations, is found by solving the simultaneous equations

$$\begin{aligned} 4x + 3y &= 12 \\ x - y &= 0 \end{aligned}$$

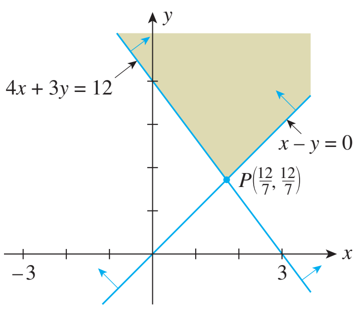


FIGURE 6
The set of points in the shaded area satisfies the system

$$\begin{aligned} 4x + 3y &\geq 12 \\ x - y &\leq 0 \end{aligned}$$

EXAMPLE 5 Sketch the solution set for the system

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + y - 6 &\leq 0 \\ 2x + y - 8 &\leq 0 \end{aligned}$$

Solution The first inequality in the system defines the right half-plane—all points to the right of the y-axis plus all points lying on the y-axis itself. The second inequality in the system defines the upper half-plane, including the x-axis. The half-planes defined by the third and fourth inequalities are indicated by arrows in Figure 7. Thus, the required region—the intersection of the four half-planes defined by the four inequalities in the given system of linear inequalities—is the shaded region. The point P is found by solving the simultaneous equations $x + y - 6 = 0$ and $2x + y - 8 = 0$.

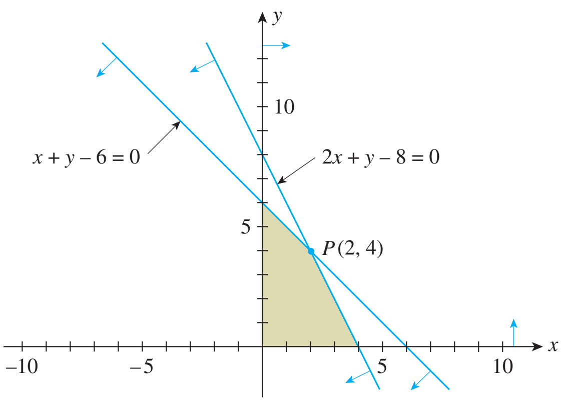


FIGURE 7
The set of points in the shaded region, including the x- and y-axes, satisfies the given inequalities.

The solution set found in Example 5 is an example of a bounded set. Observe that the set can be enclosed by a circle. For example, if you draw a circle of radius 10 with center at the origin, you will see that the set lies entirely inside the circle. On the other hand, the solution set of Example 4 cannot be enclosed by a circle and is said to be unbounded.

Bounded and Unbounded Solution Sets

The solution set of a system of linear inequalities is **bounded** if it can be enclosed by a circle. Otherwise, it is **unbounded**.

EXAMPLE 6 Determine the graphical solution set for the following system of linear inequalities:

$$\begin{aligned} 2x + y &\geq 50 \\ x + 2y &\geq 40 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

Solution The required solution set is the unbounded region shown in Figure 8.

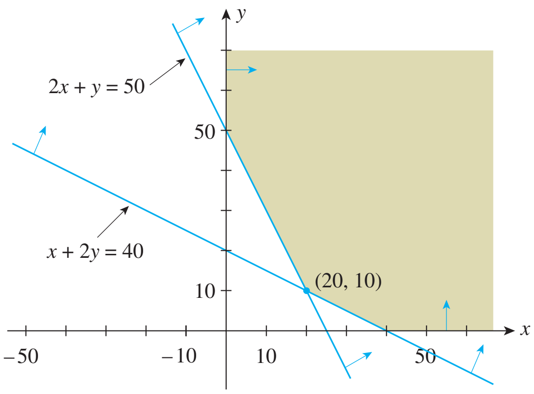


FIGURE 8
The solution set is an unbounded region.

3.1 Self-Check Exercises

1. Determine graphically the solution set for the following system of inequalities:

$$\begin{aligned} x + 2y &\le 10 \\ 5x + 3y &\le 30 \\ x \ge 0, y &\ge 0 \end{aligned}$$

2. Determine graphically the solution set for the following system of inequalities:

$$\begin{aligned} 5x + 3y &\ge 30 \\ x - 3y &\le 0 \\ x &\ge 2 \end{aligned}$$

Solutions to Self-Check Exercises 3.1 can be found on page 163.

3.1 Concept Questions

1. a. What is the difference, geometrically, between the solution set of $ax + by < c$ and the solution set of $ax + by \le c$?

b. Describe the set that is obtained by intersecting the solution set of $ax + by \le c$ with the solution set of $ax + by \ge c$.

2. a. What is the solution set of a system of linear inequalities?

b. How do you find the solution of a system of linear inequalities graphically?

3.1 Exercises

In Exercises 1–10, find the graphical solution of each inequality.

1. $4x - 8 < 0$

2. $3y + 2 > 0$

3. $x - y \le 0$

4. $3x + 4y \le -2$

5. $x \le -3$

6. $y \ge -1$

7. $2x + y \le 4$

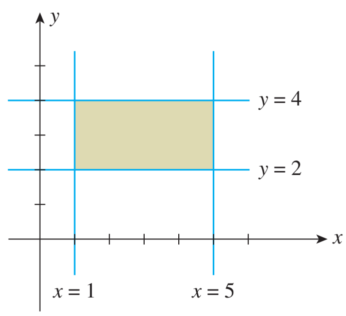
8. $-3x + 6y \ge 12$

9. $4x - 3y \le -24$

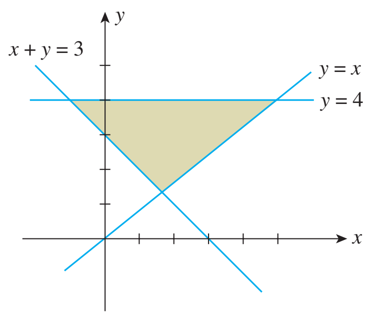
10. $5x - 3y \ge 15$

In Exercises 11–18, write a system of linear inequalities that describes the shaded region.

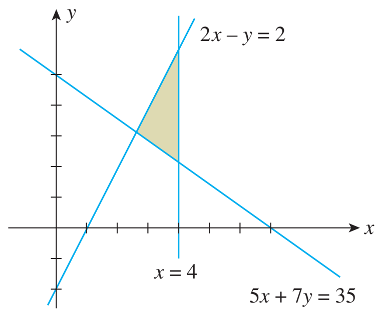
11.



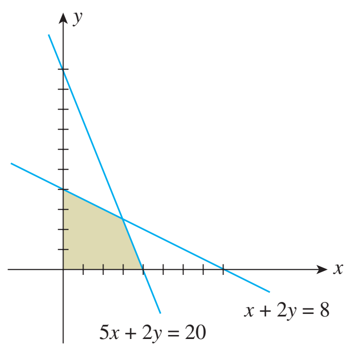
12.



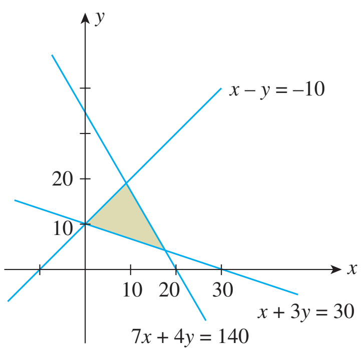
13.



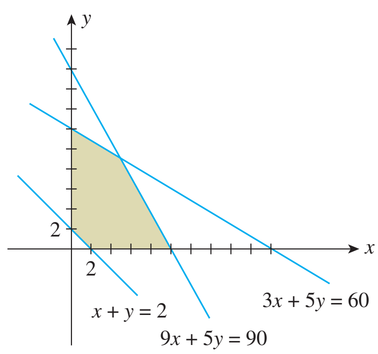
14.



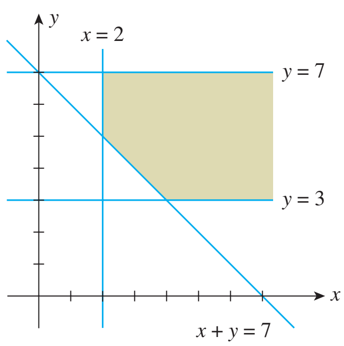
15.



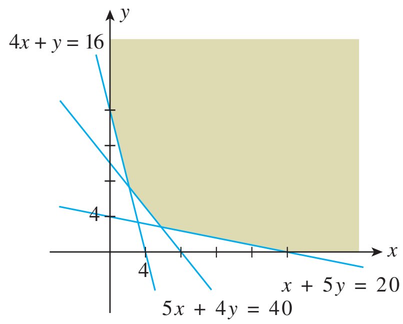
16.



17.



18.



In Exercises 19–36, determine graphically the solution set for each system of inequalities and indicate whether the solution set is bounded or unbounded.

19. $\begin{aligned} 2x + 4y &> 16 \\ -x + 3y &\geq 7 \end{aligned}$

20. $\begin{aligned} 3x - 2y &> -13 \\ -x + 2y &> 5 \end{aligned}$

21. $\begin{aligned} x - y &\leq 0 \\ 2x + 3y &\geq 10 \end{aligned}$

22. $\begin{aligned} x + y &\geq -2 \\ 3x - y &\leq 6 \end{aligned}$

23. $\begin{aligned} x + 2y &\geq 3 \\ 2x + 4y &\leq -2 \end{aligned}$

24. $\begin{aligned} 2x - y &\geq 4 \\ 4x - 2y &< -2 \end{aligned}$

25. $\begin{aligned} x + y &\leq 6 \\ 0 \leq x &\leq 3 \\ y &\geq 0 \end{aligned}$

26. $\begin{aligned} 4x - 3y &\leq 12 \\ 5x + 2y &\leq 10 \\ x \geq 0, y &\geq 0 \end{aligned}$

27. $\begin{aligned} 3x - 6y &\leq 12 \\ -x + 2y &\leq 4 \\ x \geq 0, y &\geq 0 \end{aligned}$

28. $\begin{aligned} x + y &\geq 20 \\ x + 2y &\geq 40 \\ x \geq 0, y &\geq 0 \end{aligned}$

29. $\begin{aligned} 3x - 7y &\geq -24 \\ x + 3y &\geq 8 \\ x \geq 0, y &\geq 0 \end{aligned}$

30. $\begin{aligned} 3x + 4y &\geq 12 \\ 2x - y &\geq -2 \\ 0 \leq y &\leq 3 \\ x &\geq 0 \end{aligned}$

31. $\begin{aligned} x + 2y &\geq 3 \\ 5x - 4y &\leq 16 \\ 0 \leq y &\leq 2 \\ x &\geq 0 \end{aligned}$

32. $\begin{aligned} x + y &\leq 4 \\ 2x + y &\leq 6 \\ 2x - y &\geq -1 \\ x \geq 0, y &\geq 0 \end{aligned}$

33. $\begin{aligned} 6x + 5y &\leq 30 \\ 3x + y &\geq 6 \\ x + y &\geq 4 \\ x \geq 0, y &\geq 0 \end{aligned}$

34. $\begin{aligned} 6x + 7y &\leq 84 \\ 12x - 11y &\leq 18 \\ 6x - 7y &\leq 28 \\ x \geq 0, y &\geq 0 \end{aligned}$

35. $$\begin{aligned} x - y &\geq -6 \\ x - 2y &\leq -2 \\ x + 2y &\geq 6 \\ x - 2y &\geq -14 \\ x \geq 0, y &\geq 0 \end{aligned}$$

36. $$\begin{aligned} x - 3y &\geq -18 \\ 3x - 2y &\geq 2 \\ x - 3y &\leq -4 \\ 3x - 2y &\leq 16 \\ x \geq 0, y &\geq 0 \end{aligned}$$

In Exercises 37–40, determine whether the statement is true or false. If it is true, explain why it is true. If it is false, give an example to show why it is false.

37. The solution set of a linear inequality involving two variables is either a half-plane or a straight line.

38. The solution set of the inequality $ax + by + c \leq 0$ is either a left half-plane or a lower half-plane.

39. The solution set of a system of linear inequalities in two variables is bounded if it can be enclosed by a rectangle.

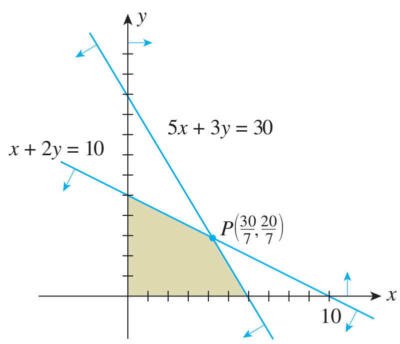
40. The solution set of the system

$$\begin{aligned} ax + by &\leq e \\ cx + dy &\leq f \\ x \geq 0, y &\geq 0 \end{aligned}$$

where a, b, c, d, e, and f are positive real numbers, is a bounded set.

3.1 Solutions to Self-Check Exercises

1. The required solution set is shown in the following figure:



The point P is found by solving the system of equations

$$\begin{aligned} x + 2y &= 10 \\ 5x + 3y &= 30 \end{aligned}$$

Solving the first equation for x in terms of y gives

$$x = 10 - 2y$$

Substituting this value of x into the second equation of the system gives

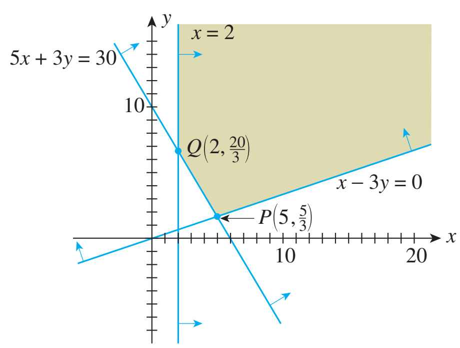
$$\begin{aligned} 5(10 - 2y) + 3y &= 30 \\ 50 - 10y + 3y &= 30 \\ -7y &= -20 \end{aligned}$$

so $y = \frac{20}{7}$. Substituting this value of y into the expression for x found earlier, we obtain

$$x = 10 - 2\left(\frac{20}{7}\right) = \frac{30}{7}$$

giving the point of intersection as $\left(\frac{30}{7}, \frac{20}{7}\right)$.

2. The required solution set is shown in the following figure:



To find the coordinates of P, we solve the system

$$\begin{aligned} 5x + 3y &= 30 \\ x - 3y &= 0 \end{aligned}$$

Solving the second equation for x in terms of y and substituting this value of x in the first equation gives

$$5(3y) + 3y = 30$$

or $y = \frac{5}{3}$. Substituting this value of y into the second equation gives $x = 5$. Next, the coordinates of Q are found by solving the system

$$\begin{aligned} 5x + 3y &= 30 \\ x &= 2 \end{aligned}$$

yielding $x = 2$ and $y = \frac{20}{3}$.

3.2 Linear Programming Problems

In many business and economic problems we are asked to optimize (maximize or minimize) a function subject to a system of equalities or inequalities. The function to be optimized is called the **objective function.** Profit functions and cost functions are examples of objective functions. The system of equalities or inequalities to which the objective function is subjected reflects the constraints (for example, limitations on resources such as materials and labor) imposed on the solution(s) to the problem. Problems of this nature are called **mathematical programming problems.** In particular, problems in which both the objective function and the constraints are expressed as linear equations or inequalities are called linear programming problems.

Linear Programming Problem

A **linear programming problem** consists of a linear objective function to be maximized or minimized subject to certain constraints in the form of linear equations or inequalities.

A Maximization Problem

As an example of a linear programming problem in which the objective function is to be maximized, let's consider the following simplified version of a production problem involving two variables.

APPLIED EXAMPLE 1 A Production Problem Ace Novelty wishes to produce two types of souvenirs: type A and type B. Each type-A souvenir will result in a profit of \$1, and each type-B souvenir will result in a profit of \$1.20. To manufacture a type-A souvenir requires 2 minutes on machine I and 1 minute on machine II. A type-B souvenir requires 1 minute on machine I and 3 minutes on machine II. There are 3 hours available on machine I and 5 hours available on machine II. How many souvenirs of each type should Ace make in order to maximize its profit?

Solution As a first step toward the mathematical formulation of this problem, we tabulate the given information (see Table 1).

TABLE 1

	Type A	Type B	Time Available
Machine I	2 min	1 min	180 min
Machine II	1 min	3 min	300 min
Profit/Unit	\$1	\$1.20	

Let x be the number of type-A souvenirs and y the number of type-B souvenirs to be made. Then, the total profit P (in dollars) is given by

$$P = x + 1.2y$$

which is the objective function to be maximized.

The total amount of time that machine I is used is given by $2x + y$ minutes and must not exceed 180 minutes. Thus, we have the inequality

$$2x + y \leq 180$$

Similarly, the total amount of time that machine II is used is $x + 3y$ minutes and cannot exceed 300 minutes, so we are led to the inequality

$$x + 3y \le 300$$

Finally, neither x nor y can be negative, so

$$\begin{aligned} x &\ge 0 \\ y &\ge 0 \end{aligned}$$

To summarize, the problem at hand is one of maximizing the objective function $P = x + 1.2y$ subject to the system of inequalities

$$\begin{aligned} 2x + y &\le 180 \\ x + 3y &\le 300 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

The solution to this problem will be completed in Example 1, Section 3.3. ■

Minimization Problems

In the following linear programming problem, the objective function is to be minimized.

VIDEO

APPLIED EXAMPLE 2 A Nutrition Problem A nutritionist advises an individual who is suffering from iron and vitamin-B deficiency to take at least 2400 milligrams (mg) of iron, 2100 mg of vitamin B_1 (thiamine), and 1500 mg of vitamin B_2 (riboflavin) over a period of time. Two vitamin pills are suitable, brand A and brand B. Each brand-A pill costs 6 cents and contains 40 mg of iron, 10 mg of vitamin B_1, and 5 mg of vitamin B_2. Each brand-B pill costs 8 cents and contains 10 mg of iron and 15 mg each of vitamins B_1 and B_2 (Table 2). What combination of pills should the individual purchase in order to meet the minimum iron and vitamin requirements at the lowest cost?

TABLE 2

	Brand A	Brand B	Minimum Requirement
Iron	40 mg	10 mg	2400 mg
Vitamin B_1	10 mg	15 mg	2100 mg
Vitamin B_2	5 mg	15 mg	1500 mg
Cost/Pill	6¢	8¢	

Solution Let x be the number of brand-A pills and y the number of brand-B pills to be purchased. The cost C (in cents) is given by

$$C = 6x + 8y$$

and is the objective function to be minimized.

The amount of iron contained in x brand-A pills and y brand-B pills is given by $40x + 10y$ mg, and this must be greater than or equal to 2400 mg. This translates into the inequality

$$40x + 10y \ge 2400$$

Similar considerations involving the minimum requirements of vitamins B_1 and B_2 lead to the inequalities

$$\begin{aligned} 10x + 15y &\geq 2100 \\ 5x + 15y &\geq 1500 \end{aligned}$$

respectively. Thus, the problem here is to minimize $C = 6x + 8y$ subject to

$$\begin{aligned} 40x + 10y &\geq 2400 \\ 10x + 15y &\geq 2100 \\ 5x + 15y &\geq 1500 \\ x \geq 0, y &\geq 0 \end{aligned}$$

The solution to this problem will be completed in Example 2, Section 3.3. ■

APPLIED EXAMPLE 3 A Transportation Problem Curtis-Roe Aviation Industries has two plants, I and II, that produce the Zephyr jet engines used in their light commercial airplanes. There are 100 units of the engines in plant I and 110 units in plant II. The engines are shipped to two of Curtis-Roe's main assembly plants, A and B. The shipping costs (in dollars) per engine from plants I and II to the main assembly plants A and B are as follows:

	To Assembly Plant	
From	**A**	**B**
Plant I	100	60
Plant II	120	70

In a certain month, assembly plant A needs 80 engines whereas assembly plant B needs 70 engines. Find how many engines should be shipped from each plant to each main assembly plant if shipping costs are to be kept to a minimum.

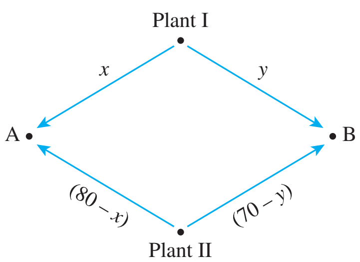


FIGURE **9**

Solution Let x denote the number of engines shipped from plant I to assembly plant A, and let y denote the number of engines shipped from plant I to assembly plant B. Since the requirements of assembly plants A and B are 80 and 70 engines, respectively, the number of engines shipped from plant II to assembly plants A and B are $(80 - x)$ and $(70 - y)$, respectively. These numbers may be displayed in a schematic. With the aid of the accompanying schematic (Figure 9) and the shipping cost schedule, we find that the total shipping cost incurred by Curtis-Roe is given by

$$\begin{aligned} C &= 100x + 60y + 120(80 - x) + 70(70 - y) \\ &= 14{,}500 - 20x - 10y \end{aligned}$$

Next, the production constraints on plants I and II lead to the inequalities

$$\begin{aligned} x + y &\leq 100 \\ (80 - x) + (70 - y) &\leq 110 \end{aligned}$$

The last inequality simplifies to

$$x + y \geq 40$$

Also, the requirements of the two main assembly plants lead to the inequalities

$$x \geq 0 \qquad y \geq 0 \qquad 80 - x \geq 0 \qquad 70 - y \geq 0$$

The last two may be written as $x \leq 80$ and $y \leq 70$.

Summarizing, we have the following linear programming problem: Minimize the objective (cost) function $C = 14{,}500 - 20x - 10y$ subject to the constraints

$$\begin{aligned} x + y &\geq 40 \\ x + y &\leq 100 \\ x &\leq 80 \\ y &\leq 70 \end{aligned}$$

where $x \geq 0$ and $y \geq 0$.

You will be asked to complete the solution to this problem in Exercise 47, Section 3.3.

APPLIED EXAMPLE 4 A Warehouse Problem Acrosonic manufactures its model F loudspeaker systems in two separate locations, plant I and plant II. The output at plant I is at most 400 per month, whereas the output at plant II is at most 600 per month. These loudspeaker systems are shipped to three warehouses that serve as distribution centers for the company. For the warehouses to meet their orders, the minimum monthly requirements of warehouses A, B, and C are 200, 300, and 400 systems, respectively. Shipping costs from plant I to warehouses A, B, and C are \$20, \$8, and \$10 per loudspeaker system, respectively, and shipping costs from plant II to each of these warehouses are \$12, \$22, and \$18, respectively. What should the shipping schedule be if Acrosonic wishes to meet the requirements of the distribution centers and at the same time keep its shipping costs to a minimum?

TABLE 3

Plant	Warehouse A	Warehouse B	Warehouse C
I	20	8	10
II	12	22	18

Solution The respective shipping costs (in dollars) per loudspeaker system may be tabulated as in Table 3. Letting x_1 denote the number of loudspeaker systems shipped from plant I to warehouse A, x_2 the number shipped from plant I to warehouse B, and so on leads to Table 4.

TABLE 4

Plant	Warehouse A	Warehouse B	Warehouse C	Max. Prod.
I	x_1	x_2	x_3	400
II	x_4	x_5	x_6	600
Min. Req.	200	300	400	

From Tables 3 and 4 we see that the cost of shipping x_1 loudspeaker systems from plant I to warehouse A is \$$20x_1$, the cost of shipping x_2 loudspeaker systems from plant I to warehouse B is \$$8x_2$, and so on. Thus, the total monthly shipping cost (in dollars) incurred by Acrosonic is given by

$$C = 20x_1 + 8x_2 + 10x_3 + 12x_4 + 22x_5 + 18x_6$$

Next, the production constraints on plants I and II lead to the inequalities

$$\begin{aligned} x_1 + x_2 + x_3 &\leq 400 \\ x_4 + x_5 + x_6 &\leq 600 \end{aligned}$$

(see Table 4). Also, the minimum requirements of each of the three warehouses lead to the three inequalities

$$\begin{aligned} x_1 + x_4 &\geq 200 \\ x_2 + x_5 &\geq 300 \\ x_3 + x_6 &\geq 400 \end{aligned}$$

Summarizing, we have the following linear programming problem:

$$\begin{aligned}
\text{Minimize} \quad & C = 20x_1 + 8x_2 + 10x_3 + 12x_4 + 22x_5 + 18x_6 \\
\text{subject to} \quad & x_1 + x_2 + x_3 \le 400 \\
& x_4 + x_5 + x_6 \le 600 \\
& x_1 + x_4 \ge 200 \\
& x_2 + x_5 \ge 300 \\
& x_3 + x_6 \ge 400 \\
& x_1 \ge 0, x_2 \ge 0, \ldots, x_6 \ge 0
\end{aligned}$$

The solution to this problem will be completed in Section 4.2, Example 5.

3.2 Self-Check Exercise

Gino Balduzzi, proprietor of Luigi's Pizza Palace, allocates \$9000 a month for advertising in two newspapers, the *City Tribune* and the *Daily News*. The *City Tribune* charges \$300 for a certain advertisement, whereas the *Daily News* charges \$100 for the same ad. Gino has stipulated that the ad is to appear in at least 15 but no more than 30 editions of the *Daily News* per month. The *City Tribune* has a daily circulation of 50,000, and the *Daily News* has a circulation of 20,000. Under these conditions, determine how many ads Gino should place in each newspaper in order to reach the largest number of readers. Formulate but do not solve the problem. (The solution to this problem can be found in Exercise 3 of Solutions to Self-Check Exercises 3.3.)

The solution to Self-Check Exercise 3.2 can be found on page 172.

3.2 Concept Questions

1. What is a linear programming problem?
2. Suppose you are asked to formulate a linear programming problem in two variables x and y. How would you express the fact that x and y are nonnegative? Why are these conditions often required in practical problems?
3. What is the difference between a maximization linear programming problem and a minimization linear programming problem?

3.2 Exercises

Formulate but do not solve each of the following exercises as a linear programming problem. You will be asked to solve these problems later.

1. **MANUFACTURING—PRODUCTION SCHEDULING** A company manufactures two products, A and B, on two machines, I and II. It has been determined that the company will realize a profit of \$3 on each unit of product A and a profit of \$4 on each unit of product B. To manufacture a unit of product A requires 6 min on machine I and 5 min on machine II. To manufacture a unit of product B requires 9 min on machine I and 4 min on machine II. There are 5 hr of machine time available on machine I and 3 hr of machine time available on machine II in each work shift. How many units of each product should be produced in each shift to maximize the company's profit?

2. **MANUFACTURING—PRODUCTION SCHEDULING** National Business Machines manufactures two models of fax machines: A and B. Each model A costs \$100 to make, and each model B costs \$150. The profits are \$30 for each model A and \$40 for each model B fax machine. If the total number of fax machines demanded per month does not exceed 2500 and the company has earmarked no more than \$600,000/month for manufacturing costs, how many units of each model should National make each month in order to maximize its monthly profit?

3. **MANUFACTURING—PRODUCTION SCHEDULING** Kane Manufacturing has a division that produces two models of fireplace grates, model A and model B. To produce each model A grate requires 3 lb of cast iron and 6 min of labor. To pro-

duce each model B grate requires 4 lb of cast iron and 3 min of labor. The profit for each model A grate is $2.00, and the profit for each model B grate is $1.50. If 1000 lb of cast iron and 20 hr of labor are available for the production of grates per day, how many grates of each model should the division produce per day in order to maximize Kane's profits?

4. **Manufacturing—Production Scheduling** Refer to Exercise 3. Because of a backlog of orders on model A grates, the manager of Kane Manufacturing has decided to produce at least 150 of these models a day. Operating under this additional constraint, how many grates of each model should Kane produce to maximize profit?

5. **Manufacturing—Production Scheduling** A division of the Winston Furniture Company manufactures dining tables and chairs. Each table requires 40 board feet of wood and 3 labor-hours. Each chair requires 16 board feet of wood and 4 labor-hours. The profit for each table is $45, and the profit for each chair is $20. In a certain week, the company has 3200 board feet of wood available, and 520 labor-hours. How many tables and chairs should Winston manufacture in order to maximize its profits?

6. **Manufacturing—Production Scheduling** Refer to Exercise 5. If the profit for each table is $50 and the profit for each chair is $18, how many tables and chairs should Winston manufacture in order to maximize its profits?

7. **Finance—Allocation of Funds** Madison Finance has a total of $20 million earmarked for homeowner and auto loans. On the average, homeowner loans have a 10% annual rate of return whereas auto loans yield a 12% annual rate of return. Management has also stipulated that the total amount of homeowner loans should be greater than or equal to 4 times the total amount of automobile loans. Determine the total amount of loans of each type Madison should extend to each category in order to maximize its returns.

8. **Investments—Asset Allocation** A financier plans to invest up to $500,000 in two projects. Project A yields a return of 10% on the investment whereas project B yields a return of 15% on the investment. Because the investment in project B is riskier than the investment in project A, the financier has decided that the investment in project B should not exceed 40% of the total investment. How much should she invest in each project in order to maximize the return on her investment?

9. **Manufacturing—Production Scheduling** Acoustical Company manufactures a CD storage cabinet that can be bought fully assembled or as a kit. Each cabinet is processed in the fabrications department and the assembly department. If the fabrication department only manufactures fully assembled cabinets, then it can produce 200 units/day; and if it only manufactures kits, it can produce 200 units/day. If the assembly department only produces fully assembled cabinets, then it can produce 100 units/day; but if it only produces kits, then it can produce 300 units/day. Each fully assembled cabinet contributes $50 to the profits of the company whereas each kit contributes $40 to its profits. How many fully assembled units and how many kits should the company produce per day in order to maximize its profits?

10. **Agriculture—Crop Planning** A farmer plans to plant two crops, A and B. The cost of cultivating crop A is $40/acre whereas that of crop B is $60/acre. The farmer has a maximum of $7400 available for land cultivation. Each acre of crop A requires 20 labor-hours, and each acre of crop B requires 25 labor-hours. The farmer has a maximum of 3300 labor-hours available. If she expects to make a profit of $150/acre on crop A and $200/acre on crop B, how many acres of each crop should she plant in order to maximize her profit?

11. **Mining—Production** Perth Mining Company operates two mines for the purpose of extracting gold and silver. The Saddle Mine costs $14,000/day to operate, and it yields 50 oz of gold and 3000 oz of silver each day. The Horseshoe Mine costs $16,000/day to operate, and it yields 75 oz of gold and 1000 oz of silver each day. Company management has set a target of at least 650 oz of gold and 18,000 oz of silver. How many days should each mine be operated so that the target can be met at a minimum cost?

12. **Transportation** Deluxe River Cruises operates a fleet of river vessels. The fleet has two types of vessels: A type-A vessel has 60 deluxe cabins and 160 standard cabins, whereas a type-B vessel has 80 deluxe cabins and 120 standard cabins. Under a charter agreement with Odyssey Travel Agency, Deluxe River Cruises is to provide Odyssey with a minimum of 360 deluxe and 680 standard cabins for their 15-day cruise in May. It costs $44,000 to operate a type-A vessel and $54,000 to operate a type-B vessel for that period. How many of each type vessel should be used in order to keep the operating costs to a minimum?

13. **Water Supply** The water-supply manager for a Midwest city needs to supply the city with at least 10 million gal of potable (drinkable) water per day. The supply may be drawn from the local reservoir or from a pipeline to an adjacent town. The local reservoir has a maximum daily yield of 5 million gallons of potable water, and the pipeline has a maximum daily yield of 10 million gallons. By contract, the pipeline is required to supply a minimum of 6 million gallons/day. If the cost for 1 million gallons of reservoir water is $300 and that for pipeline water is $500, how much water should the manager get from each source to minimize daily water costs for the city?

14. **Manufacturing—Production Scheduling** Ace Novelty manufactures "Giant Pandas" and "Saint Bernards." Each Panda requires 1.5 yd^2 of plush, 30 ft^3 of stuffing, and 5 pieces of trim; each Saint Bernard requires 2 yd^2 of plush, 35 ft^3 of stuffing, and 8 pieces of trim. The profit for each Panda is $10 and the profit for each Saint Bernard is $15. If 3600 yd^2 of plush, 66,000 ft^3 of stuffing and 13,600 pieces of trim are available, how many of each of the stuffed animals should the company manufacture to maximize profit?

15. **Nutrition—Diet Planning** A nutritionist at the Medical Center has been asked to prepare a special diet for certain patients. She has decided that the meals should contain a minimum of 400 mg of calcium, 10 mg of iron, and 40 mg of vitamin C. She has further decided that the meals are to be prepared from foods A and B. Each ounce of food A contains 30 mg of calcium, 1 mg of iron, 2 mg of vitamin C, and 2 mg of cholesterol. Each ounce of food B contains 25 mg of calcium, 0.5 mg of iron, 5 mg of vitamin C, and 5 mg of cholesterol. Find how many ounces of each type of food should be used in a meal so that the cholesterol content is minimized and the minimum requirements of calcium, iron, and vitamin C are met.

16. **Social Programs Planning** AntiFam, a hunger-relief organization, has earmarked between \$2 and \$2.5 million (inclusive) for aid to two African countries, country A and country B. Country A is to receive between \$1 million and \$1.5 million (inclusive), and country B is to receive at least \$0.75 million. It has been estimated that each dollar spent in country A will yield an effective return of \$.60, whereas a dollar spent in country B will yield an effective return of \$.80. How should the aid be allocated if the money is to be utilized most effectively according to these criteria?
Hint: If x and y denote the amount of money to be given to country A and country B, respectively, then the objective function to be maximized is $P = 0.6x + 0.8y$.

17. **Advertising** Everest Deluxe World Travel has decided to advertise in the Sunday editions of two major newspapers in town. These advertisements are directed at three groups of potential customers. Each advertisement in newspaper I is seen by 70,000 group-A customers, 40,000 group-B customers, and 20,000 group-C customers. Each advertisement in newspaper II is seen by 10,000 group-A, 20,000 group-B, and 40,000 group-C customers. Each advertisement in newspaper I costs \$1000, and each advertisement in newspaper II costs \$800. Everest would like their advertisements to be read by at least 2 million people from group A, 1.4 million people from group B, and 1 million people from group C. How many advertisements should Everest place in each newspaper to achieve its advertisement goals at a minimum cost?

18. **Manufacturing—Shipping Costs** TMA manufactures 37-in. high-definition LCD televisions in two separate locations, location I and location II. The output at location I is at most 6000 televisions/month, whereas the output at location II is at most 5000 televisions/month. TMA is the main supplier of televisions to Pulsar Corporation, its holding company, which has priority in having all its requirements met. In a certain month, Pulsar placed orders for 3000 and 4000 televisions to be shipped to two of its factories located in city A and city B, respectively. The shipping costs (in dollars) per television from the two TMA plants to the two Pulsar factories are as follows:

	To Pulsar Factories	
From TMA	**City A**	**City B**
Location I	\$6	\$4
Location II	\$8	\$10

Find a shipping schedule that meets the requirements of both companies while keeping costs to a minimum.

19. **Investments—Asset Allocation** A financier plans to invest up to \$2 million in three projects. She estimates that project A will yield a return of 10% on her investment, project B will yield a return of 15% on her investment, and project C will yield a return of 20% on her investment. Because of the risks associated with the investments, she decided to put not more than 20% of her total investment in project C. She also decided that her investments in projects B and C should not exceed 60% of her total investment. Finally, she decided that her investment in project A should be at least 60% of her investments in projects B and C. How much should the financier invest in each project if she wishes to maximize the total returns on her investments?

20. **Investments—Asset Allocation** Ashley has earmarked at most \$250,000 for investment in three mutual funds: a money market fund, an international equity fund, and a growth-and-income fund. The money market fund has a rate of return of 6%/year, the international equity fund has a rate of return of 10%/year, and the growth-and-income fund has a rate of return of 15%/year. Ashley has stipulated that no more than 25% of her total portfolio should be in the growth-and-income fund and that no more than 50% of her total portfolio should be in the international equity fund. To maximize the return on her investment, how much should Ashley invest in each type of fund?

21. **Manufacturing—Production Scheduling** A company manufactures products A, B, and C. Each product is processed in three departments: I, II, and III. The total available labor-hours per week for departments I, II, and III are 900, 1080, and 840, respectively. The time requirements (in hours per unit) and profit per unit for each product are as follows:

	Product A	Product B	Product C
Dept. I	2	1	2
Dept. II	3	1	2
Dept. III	2	2	1
Profit	\$18	\$12	\$15

How many units of each product should the company produce in order to maximize its profit?

22. **Advertising** As part of a campaign to promote its annual clearance sale, the Excelsior Company decided to buy television advertising time on Station KAOS. Excelsior's advertising budget is \$102,000. Morning time costs \$3000/minute, afternoon time costs \$1000/minute, and evening (prime) time costs \$12,000/minute. Because of previous commitments, KAOS cannot offer Excelsior more than 6 min of prime time or more than a total of 25 min of advertising time over the 2 weeks in which the commercials are to be run. KAOS estimates that morning commercials are seen by 200,000 people, afternoon commercials are seen by 100,000 people, and evening com-

mercials are seen by 600,000 people. How much morning, afternoon, and evening advertising time should Excelsior buy in order to maximize exposure of its commercials?

23. **Manufacturing—Production Scheduling** Custom Office Furniture Company is introducing a new line of executive desks made from a specially selected grade of walnut. Initially, three different models—A, B, and C—are to be marketed. Each model A desk requires $1\frac{1}{4}$ hr for fabrication, 1 hr for assembly, and 1 hr for finishing; each model B desk requires $1\frac{1}{2}$ hr for fabrication, 1 hr for assembly, and 1 hr for finishing; each model C desk requires $1\frac{1}{2}$ hr, $\frac{3}{4}$ hr, and $\frac{1}{2}$ hr for fabrication, assembly, and finishing, respectively. The profit on each model A desk is \$26, the profit on each model B desk is \$28, and the profit on each model C desk is \$24. The total time available in the fabrication department, the assembly department, and the finishing department in the first month of production is 310 hr, 205 hr, and 190 hr, respectively. To maximize Custom's profit, how many desks of each model should be made in the month?

24. **Manufacturing—Shipping Costs** Acrosonic of Example 4 also manufactures a model G loudspeaker system in plants I and II. The output at plant I is at most 800 systems/month whereas the output at plant II is at most 600/month. These loudspeaker systems are also shipped to the three warehouses—A, B, and C—whose minimum monthly requirements are 500, 400, and 400, respectively. Shipping costs from plant I to warehouse A, warehouse B, and warehouse C are \$16, \$20, and \$22 per system, respectively, and shipping costs from plant II to each of these warehouses are \$18, \$16, and \$14 per system, respectively. What shipping schedule will enable Acrosonic to meet the warehouses' requirements and at the same time keep its shipping costs to a minimum?

25. **Manufacturing—Shipping Costs** Steinwelt Piano manufactures uprights and consoles in two plants, plant I and plant II. The output of plant I is at most 300/month, whereas the output of plant II is at most 250/month. These pianos are shipped to three warehouses that serve as distribution centers for the company. To fill current and projected future orders, warehouse A requires a minimum of 200 pianos/month, warehouse B requires at least 150 pianos/month, and warehouse C requires at least 200 pianos/month. The shipping cost of each piano from plant I to warehouse A, warehouse B, and warehouse C is \$60, \$60, and \$80, respectively, and the shipping cost of each piano from plant II to warehouse A, warehouse B, and warehouse C is \$80, \$70, and \$50, respectively. What shipping schedule will enable Steinwelt to meet the warehouses' requirements while keeping shipping costs to a minimum?

26. **Manufacturing—Prefabricated Housing Production** Boise Lumber has decided to enter the lucrative prefabricated housing business. Initially, it plans to offer three models: standard, deluxe, and luxury. Each house is prefabricated and partially assembled in the factory, and the final assembly is completed on site. The dollar amount of building material required, the amount of labor required in the factory for prefabrication and partial assembly, the amount of on-site labor required, and the profit per unit are as follows:

	Standard Model	Deluxe Model	Luxury Model
Material	\$6,000	\$8,000	\$10,000
Factory Labor (hr)	240	220	200
On-site Labor (hr)	180	210	300
Profit	\$3,400	\$4,000	\$5,000

For the first year's production, a sum of \$8.2 million is budgeted for the building material; the number of labor-hours available for work in the factory (for prefabrication and partial assembly) is not to exceed 218,000 hr; and the amount of labor for on-site work is to be less than or equal to 237,000 labor-hours. Determine how many houses of each type Boise should produce (market research has confirmed that there should be no problems with sales) in order to maximize its profit from this new venture.

27. **Production—Juice Products** CalJuice Company has decided to introduce three fruit juices made from blending two or more concentrates. These juices will be packaged in 2-qt (64-oz) cartons. One carton of pineapple–orange juice requires 8 oz each of pineapple and orange juice concentrates. One carton of orange–banana juice requires 12 oz of orange juice concentrate and 4 oz of banana pulp concentrate. Finally, one carton of pineapple–orange–banana juice requires 4 oz of pineapple juice concentrate, 8 oz of orange juice concentrate, and 4 oz of banana pulp. The company has decided to allot 16,000 oz of pineapple juice concentrate, 24,000 oz of orange juice concentrate, and 5000 oz of banana pulp concentrate for the initial production run. The company has also stipulated that the production of pineapple–orange–banana juice should not exceed 800 cartons. Its profit on one carton of pineapple–orange juice is \$1.00, its profit on one carton of orange–banana juice is \$.80, and its profit on one carton of pineapple–orange–banana juice is \$.90. To realize a maximum profit, how many cartons of each blend should the company produce?

28. **Manufacturing—Cold Formula Production** Beyer Pharmaceutical produces three kinds of cold formulas: formula I, formula II, and formula III. It takes 2.5 hr to produce 1000 bottles of formula I, 3 hr to produce 1000 bottles of formula II, and 4 hr to produce 1000 bottles of formula III. The profits for each 1000 bottles of formula I, formula II, and formula III are \$180, \$200, and \$300, respectively. For a certain production run, there are enough ingredients on hand to make at most 9000 bottles of formula I, 12,000 bottles of formula II, and 6000 bottles of formula III. Furthermore, the time for the production run is limited to a maximum of 70 hr. How many bottles of each formula should be produced in this production run so that the profit is maximized?

In Exercises 29 and 30, determine whether the statement is true or false. If it is true, explain why it is true. If it is false, give an example to show why it is false.

29. The problem

$$\begin{aligned} \text{Maximize} \quad & P = xy \\ \text{subject to} \quad & 2x + 3y \leq 12 \\ & 2x + y \leq 8 \\ & x \geq 0, y \geq 0 \end{aligned}$$

is a linear programming problem.

30. The problem

$$\begin{aligned} \text{Minimize} \quad & C = 2x + 3y \\ \text{subject to} \quad & 2x + 3y \leq 6 \\ & x - y = 0 \\ & x \geq 0, y \geq 0 \end{aligned}$$

is a linear programming problem.

3.2 Solution to Self-Check Exercise

Let x denote the number of ads to be placed in the *City Tribune* and y the number to be placed in the *Daily News*. The total cost for placing x ads in the *City Tribune* and y ads in the *Daily News* is $300x + 100y$ dollars, and since the monthly budget is $9000, we must have

$$300x + 100y \leq 9000$$

Next, the condition that the ad must appear in at least 15 but no more than 30 editions of the *Daily News* translates into the inequalities

$$\begin{aligned} y &\geq 15 \\ y &\leq 30 \end{aligned}$$

Finally, the objective function to be maximized is

$$P = 50{,}000x + 20{,}000y$$

To summarize, we have the following linear programming problem:

$$\begin{aligned} \text{Maximize} \quad & P = 50{,}000x + 20{,}000y \\ \text{subject to} \quad & 300x + 100y \leq 9000 \\ & y \geq 15 \\ & y \leq 30 \\ & x \geq 0, y \geq 0 \end{aligned}$$

3.3 Graphical Solution of Linear Programming Problems

The Graphical Method

Linear programming problems in two variables have relatively simple geometric interpretations. For example, the system of linear constraints associated with a two-dimensional linear programming problem, unless it is inconsistent, defines a planar region or a line segment whose boundary is composed of straight-line segments and/or half-lines. Such problems are therefore amenable to graphical analysis.

Consider the following two-dimensional linear programming problem:

$$\begin{aligned} \text{Maximize} \quad & P = 3x + 2y \\ \text{subject to} \quad & 2x + 3y \leq 12 \\ & 2x + y \leq 8 \\ & x \geq 0, y \geq 0 \end{aligned} \qquad \textbf{(7)}$$

The system of linear inequalities in (7) defines the planar region S shown in Figure 10. Each point in S is a candidate for the solution of the problem at hand and is referred to as a **feasible solution**. The set S itself is referred to as a **feasible set**. Our goal is to find, from among all the points in the set S, the point(s) that optimizes the objective function P. Such a feasible solution is called an **optimal solution** and constitutes the solution to the linear programming problem under consideration.

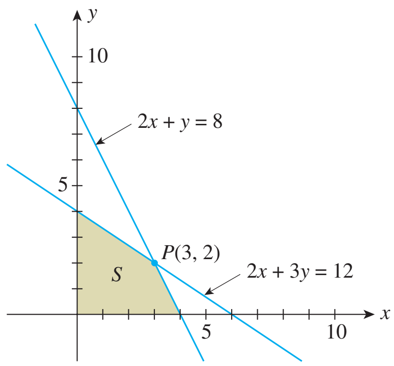


FIGURE **10**
Each point in the feasible set S is a candidate for the optimal solution.

As noted earlier, each point $P(x, y)$ in S is a candidate for the optimal solution to the problem at hand. For example, the point (1, 3) is easily seen to lie in S and is therefore in the running. The value of the objective function P at the point (1, 3) is given by $P = 3(1) + 2(3) = 9$. Now, if we could compute the value of P corresponding to each point in S, then the point(s) in S that gave the largest value to P would constitute the solution set sought. Unfortunately, in most problems the number of candidates either is too large or, as in this problem, is infinite. Thus, this method is at best unwieldy and at worst impractical.

Let's turn the question around. Instead of asking for the value of the objective function P at a feasible point, let's assign a value to the objective function P and ask whether there are feasible points that would correspond to the given value of P. Toward this end, suppose we assign a value of 6 to P. Then the objective function P becomes $3x + 2y = 6$, a linear equation in x and y, and thus it has a graph that is a straight line L_1 in the plane. In Figure 11, we have drawn the graph of this straight line superimposed on the feasible set S.

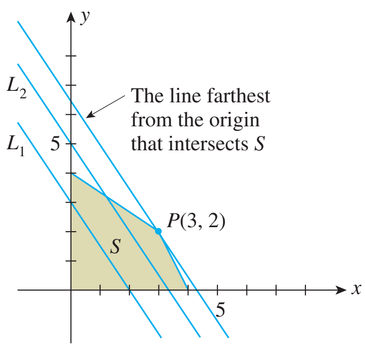


FIGURE **11**
A family of parallel lines that intersect the feasible set S

It is clear that each point on the straight-line segment given by the intersection of the straight line L_1 and the feasible set S corresponds to the given value, 6, of P. For this reason the line L_1 is called an **isoprofit line.** Let's repeat the process, this time assigning a value of 10 to P. We obtain the equation $3x + 2y = 10$ and the line L_2 (see Figure 11), which suggests that there are feasible points that correspond to a larger value of P. Observe that the line L_2 is parallel to the line L_1 because both lines have slope equal to $-\frac{3}{2}$, which is easily seen by casting the corresponding equations in the slope-intercept form.

In general, by assigning different values to the objective function, we obtain a family of parallel lines, each with slope equal to $-\frac{3}{2}$. Furthermore, a line corresponding to a larger value of P lies farther away from the origin than one with a smaller value of P. The implication is clear. To obtain the optimal solution(s) to the problem at hand, find the straight line, from this family of straight lines, that is farthest from the origin and still intersects the feasible set S. The required line is the one that passes through the point $P(3, 2)$ (see Figure 11), so the solution to the problem is given by $x = 3$, $y = 2$, resulting in a maximum value of $P = 3(3) + 2(2) = 13$.

That the optimal solution to this problem was found to occur at a vertex of the feasible set S is no accident. In fact, the result is a consequence of the following basic theorem on linear programming, which we state without proof.

THEOREM 1
Linear Programming

If a linear programming problem has a solution then it must occur at a vertex, or corner point, of the feasible set S associated with the problem.

Furthermore, if the objective function P is optimized at two adjacent vertices of S, then it is optimized at every point on the line segment joining these vertices, in which case there are infinitely many solutions to the problem.

Theorem 1 tells us that our search for the solution(s) to a linear programming problem may be restricted to the examination of the set of vertices of the feasible set S associated with the problem. Since a feasible set S has finitely many vertices, the theorem suggests that the solution(s) to the linear programming problem may be found by inspecting the values of the objective function P at these vertices.

Although Theorem 1 sheds some light on the nature of the solution of a linear programming problem, it does not tell us when a linear programming problem has a solution. The following theorem states some conditions that guarantee when a linear programming problem has a solution.

THEOREM 2
Existence of a Solution

Suppose we are given a linear programming problem with a feasible set S and an objective function $P = ax + by$.

a. If S is bounded, then P has both a maximum and a minimum value on S.

b. If S is unbounded and both a and b are nonnegative, then P has a minimum value on S provided that the constraints defining S include the inequalities $x \geq 0$ and $y \geq 0$.

c. If S is the empty set, then the linear programming problem has no solution; that is, P has neither a maximum nor a minimum value.

The **method of corners**, a simple procedure for solving linear programming problems based on Theorem 1, follows.

The Method of Corners

1. Graph the feasible set.
2. Find the coordinates of all corner points (vertices) of the feasible set.
3. Evaluate the objective function at each corner point.
4. Find the vertex that renders the objective function a maximum (minimum). If there is only one such vertex, then this vertex constitutes a unique solution to the problem. If the objective function is maximized (minimized) at two adjacent corner points of S, there are infinitely many optimal solutions given by the points on the line segment determined by these two vertices.

VIDEO

APPLIED EXAMPLE 1 Maximizing Profit We are now in a position to complete the solution to the production problem posed in Example 1, Section 3.2. Recall that the mathematical formulation led to the following linear programming problem:

$$\begin{aligned} \text{Maximize} \quad & P = x + 1.2y \\ \text{subject to} \quad & 2x + y \le 180 \\ & x + 3y \le 300 \\ & x \ge 0, y \ge 0 \end{aligned}$$

Solution The feasible set S for the problem is shown in Figure 12.

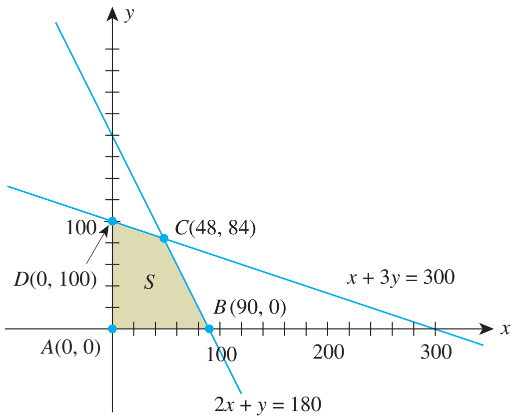


FIGURE **12**
The corner point that yields the maximum profit is $C(48, 84)$.

The vertices of the feasible set are $A(0, 0)$, $B(90, 0)$, $C(48, 84)$, and $D(0, 100)$. The values of P at these vertices may be tabulated as follows:

Vertex	$P = x + 1.2y$
$A(0, 0)$	0
$B(90, 0)$	90
$C(48, 84)$	148.8
$D(0, 100)$	120

From the table, we see that the maximum of $P = x + 1.2y$ occurs at the vertex (48, 84) and has a value of 148.8. Recalling what the symbols x, y, and P represent, we conclude that Ace Novelty would maximize its profit (a figure of \$148.80) by producing 48 type-A souvenirs and 84 type-B souvenirs.

Explore & Discuss

Consider the linear programming problem

$$\begin{aligned} \text{Maximize} \quad & P = 4x + 3y \\ \text{subject to} \quad & 2x + y \le 10 \\ & 2x + 3y \le 18 \\ & x \ge 0, y \ge 0 \end{aligned}$$

1. Sketch the feasible set S for the linear programming problem.
2. Draw the isoprofit lines superimposed on S corresponding to $P = 12$, 16, 20, and 24, and show that these lines are parallel to each other.
3. Show that the solution to the linear programming problem is $x = 3$ and $y = 4$. Is this result the same as that found using the method of corners?

VIDEO

APPLIED EXAMPLE 2 A Nutrition Problem Complete the solution of the nutrition problem posed in Example 2, Section 3.2.

Solution Recall that the mathematical formulation of the problem led to the following linear programming problem in two variables:

$$\begin{aligned} \text{Minimize} \quad & C = 6x + 8y \\ \text{subject to} \quad & 40x + 10y \geq 2400 \\ & 10x + 15y \geq 2100 \\ & 5x + 15y \geq 1500 \\ & x \geq 0, y \geq 0 \end{aligned}$$

The feasible set S defined by the system of constraints is shown in Figure 13.

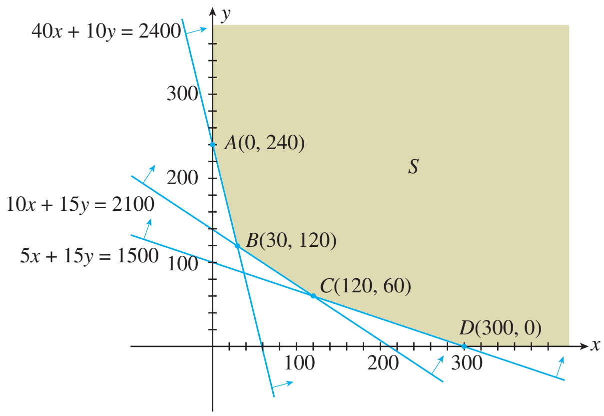


FIGURE 13
The corner point that yields the minimum cost is $B(30, 120)$.

The vertices of the feasible set S are $A(0, 240)$, $B(30, 120)$, $C(120, 60)$, and $D(300, 0)$. The values of the objective function C at these vertices are given in the following table:

Vertex	$C = 6x + 8y$
$A(0, 240)$	1920
$B(30, 120)$	1140
$C(120, 60)$	1200
$D(300, 0)$	1800

From the table, we can see that the minimum for the objective function $C = 6x + 8y$ occurs at the vertex $B(30, 120)$ and has a value of 1140. Thus, the individual should purchase 30 brand-A pills and 120 brand-B pills at a minimum cost of \$11.40.

EXAMPLE 3 A Linear Programming Problem with Multiple Solutions Find the maximum and minimum of $P = 2x + 3y$ subject to the following system of linear inequalities:

$$\begin{aligned} 2x + 3y &\leq 30 \\ -x + y &\leq 5 \\ x + y &\geq 5 \\ x &\leq 10 \\ x \geq 0, y &\geq 0 \end{aligned}$$

Solution The feasible set S is shown in Figure 14. The vertices of the feasible set S are $A(5, 0)$, $B(10, 0)$, $C(10, \frac{10}{3})$, $D(3, 8)$, and $E(0, 5)$. The values of the objective function P at these vertices are given in the following table:

Vertex	$P = 2x + 3y$
$A(5, 0)$	10
$B(10, 0)$	20
$C(10, \frac{10}{3})$	30
$D(3, 8)$	30
$E(0, 5)$	15

From the table, we see that the maximum for the objective function $P = 2x + 3y$ occurs at the vertices $C(10, \frac{10}{3})$ and $D(3, 8)$. This tells us that every point on the line segment joining the points $C(10, \frac{10}{3})$ and $D(3, 8)$ maximizes P, giving it a value of 30 at each of these points. From the table, it is also clear that P is minimized at the point $(5, 0)$, where it attains a value of 10.

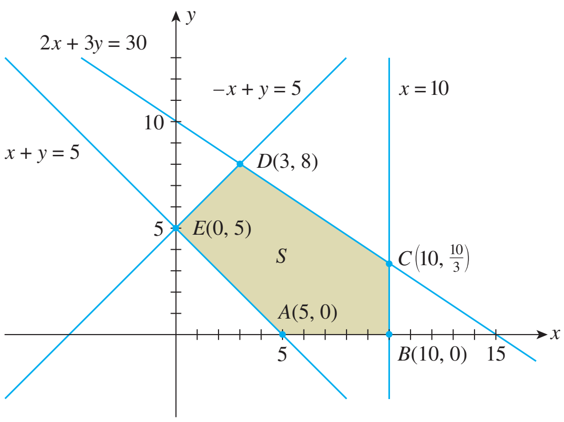


FIGURE 14
Every point lying on the line segment joining C and D maximizes P.

Explore & Discuss

Consider the linear programming problem

$$\begin{aligned} \text{Maximize} \quad & P = 2x + 3y \\ \text{subject to} \quad & 2x + y \leq 10 \\ & 2x + 3y \leq 18 \\ & x \geq 0, y \geq 0 \end{aligned}$$

1. Sketch the feasible set S for the linear programming problem.
2. Draw the isoprofit lines superimposed on S corresponding to $P = 6$, 8, 12, and 18, and show that these lines are parallel to each other.
3. Show that there are infinitely many solutions to the problem. Is this result as predicted by the method of corners?

We close this section by examining two situations in which a linear programming problem may have no solution.

EXAMPLE 4 An Unbounded Linear Programming Problem with No Solution Solve the following linear programming problem:

$$\begin{array}{ll} \text{Maximize} & P = x + 2y \\ \text{subject to} & -2x + y \le 4 \\ & x - 3y \le 3 \\ & x \ge 0, y \ge 0 \end{array}$$

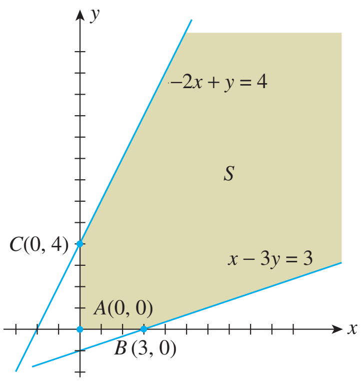


FIGURE 15
This maximization problem has no solution because the feasible set is unbounded.

Solution The feasible set S for this problem is shown in Figure 15. Since the set S is unbounded (both x and y can take on arbitrarily large positive values), we see that we can make P as large as we please by making x and y large enough. This problem has no solution. The problem is said to be unbounded. ■

EXAMPLE 5 An Infeasible Linear Programming Problem Solve the following linear programming problem:

$$\begin{array}{ll} \text{Maximize} & P = x + 2y \\ \text{subject to} & x + 2y \le 4 \\ & 2x + 3y \ge 12 \\ & x \ge 0, y \ge 0 \end{array}$$

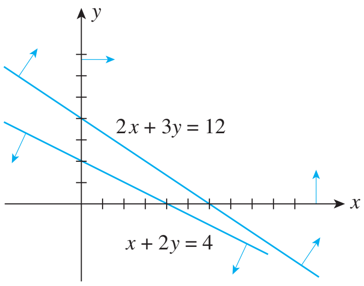


FIGURE 16
This problem is inconsistent because there is no point that satisfies all of the given inequalities.

Solution The half-planes described by the constraints (inequalities) have no points in common (Figure 16). Hence there are no feasible points and the problem has no solution. In this situation, we say that the problem is **infeasible,** or **inconsistent.** ■

The situations described in Examples 5 and 6 are unlikely to occur in well-posed problems arising from practical applications of linear programming.

The method of corners is particularly effective in solving two-variable linear programming problems with a small number of constraints, as the preceding examples have amply demonstrated. Its effectiveness, however, decreases rapidly as the number of variables and/or constraints increases. For example, it may be shown that a linear programming problem in three variables and five constraints may have up to ten feasible corner points. The determination of the feasible corner points calls for the solution of ten 3×3 systems of linear equations and then the verification—by the substitution of each of these solutions into the system of constraints—to see if it is, in fact, a feasible point. When the number of variables and constraints goes up to five and ten, respectively (still a very small system from the standpoint of applications in economics), the number of vertices to be found and checked for feasible corner points increases dramatically to 252, and each of these vertices is found by solving a 5×5 linear system! For this reason, the method of corners is seldom used to solve linear programming problems; its redeeming value lies in the fact that much insight is gained into the nature of the solutions of linear programming problems through its use in solving two-variable problems.

3.3 Self-Check Exercises

1. Use the method of corners to solve the following linear programming problem:

$$\begin{array}{ll} \text{Maximize} & P = 4x + 5y \\ \text{subject to} & x + 2y \le 10 \\ & 5x + 3y \le 30 \\ & x \ge 0, y \ge 0 \end{array}$$

2. Use the method of corners to solve the following linear programming problem:

$$\begin{array}{ll} \text{Minimize} & C = 5x + 3y \\ \text{subject to} & 5x + 3y \ge 30 \\ & x - 3y \le 0 \\ & x \ge 2 \end{array}$$

3. Gino Balduzzi, proprietor of Luigi's Pizza Palace, allocates \$9000 a month for advertising in two newspapers, the *City Tribune* and the *Daily News*. The *City Tribune* charges \$300 for a certain advertisement, whereas the *Daily News* charges \$100 for the same ad. Gino has stipulated that the ad is to appear in at least 15 but no more than 30 editions of the *Daily News* per month. The *City Tribune* has a daily circulation of 50,000, and the *Daily News* has a circulation of 20,000. Under these conditions, determine how many ads Gino should place in each newspaper in order to reach the largest number of readers.

Solutions to Self-Check Exercises 3.3 can be found on page 184.

3.3 Concept Questions

1. a. What is the feasible set associated with a linear programming problem?

b. What is a feasible solution of a linear programming problem?

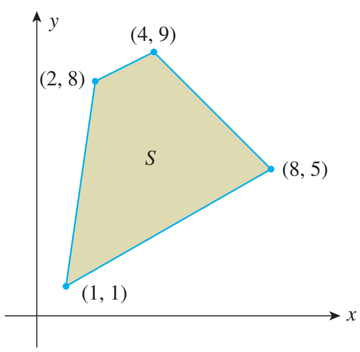
c. What is an optimal solution of a linear programming problem?

2. Describe the method of corners.

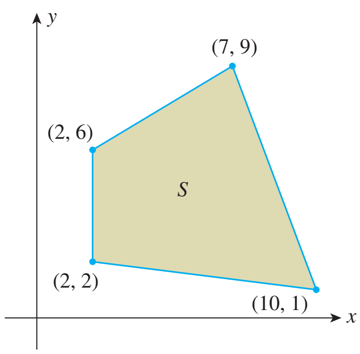
3.3 Exercises

In Exercises 1–6, find the maximum and/or minimum value(s) of the objective function on the feasible set *S*.

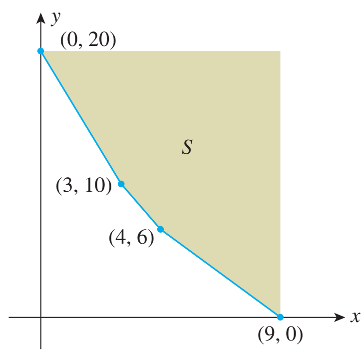
1. $Z = 2x + 3y$



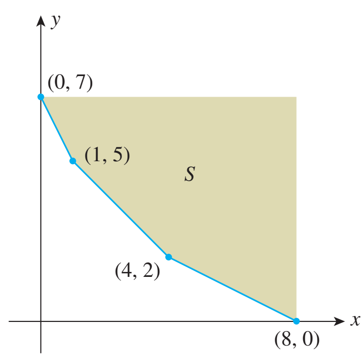
2. $Z = 3x - y$



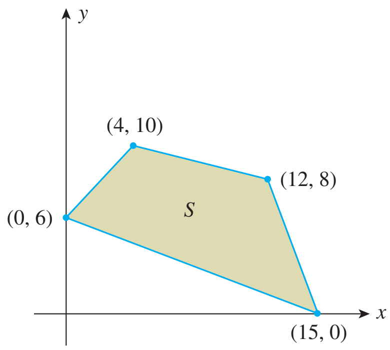
3. $Z = 3x + 4y$



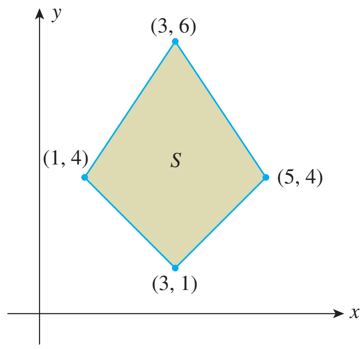
4. $Z = 7x + 9y$



5. $Z = x + 4y$



6. $Z = 3x + 2y$



In Exercises 7–28, solve each linear programming problem by the method of corners.

7. Maximize $P = 2x + 3y$
subject to
$$\begin{aligned} x + y &\le 6 \\ x &\le 3 \\ x \ge 0, y &\ge 0 \end{aligned}$$

8. Maximize $P = x + 2y$
subject to
$$\begin{aligned} x + y &\le 4 \\ 2x + y &\le 5 \\ x \ge 0, y &\ge 0 \end{aligned}$$

9. Maximize $P = 2x + y$ subject to the constraints of Exercise 8.

10. Maximize $P = 4x + 2y$
subject to
$$\begin{aligned} x + y &\le 8 \\ 2x + y &\le 10 \\ x \ge 0, y &\ge 0 \end{aligned}$$

11. Maximize $P = x + 8y$ subject to the constraints of Exercise 10.

12. Maximize $P = 3x - 4y$
subject to
$$\begin{aligned} x + 3y &\le 15 \\ 4x + y &\le 16 \\ x \ge 0, y &\ge 0 \end{aligned}$$

13. Maximize $P = x + 3y$
subject to
$$\begin{aligned} 2x + y &\le 6 \\ x + y &\le 4 \\ x &\le 1 \\ x \ge 0, y &\ge 0 \end{aligned}$$

14. Maximize $P = 2x + 5y$
subject to
$$\begin{aligned} 2x + y &\le 16 \\ 2x + 3y &\le 24 \\ y &\le 6 \\ x \ge 0, y &\ge 0 \end{aligned}$$

15. Minimize $C = 3x + 4y$
subject to
$$\begin{aligned} x + y &\ge 3 \\ x + 2y &\ge 4 \\ x \ge 0, y &\ge 0 \end{aligned}$$

16. Minimize $C = 2x + 4y$ subject to the constraints of Exercise 15.

17. Minimize $C = 3x + 6y$
subject to
$$\begin{aligned} x + 2y &\ge 40 \\ x + y &\ge 30 \\ x \ge 0, y &\ge 0 \end{aligned}$$

18. Minimize $C = 3x + y$ subject to the constraints of Exercise 17.

19. Minimize $C = 2x + 10y$
subject to
$$\begin{aligned} 5x + 2y &\ge 40 \\ x + 2y &\ge 20 \\ y \ge 3, x &\ge 0 \end{aligned}$$

20. Minimize $C = 2x + 5y$
subject to
$$\begin{aligned} 4x + y &\ge 40 \\ 2x + y &\ge 30 \\ x + 3y &\ge 30 \\ x \ge 0, y &\ge 0 \end{aligned}$$

21. Minimize $C = 10x + 15y$
subject to
$$\begin{aligned} x + y &\le 10 \\ 3x + y &\ge 12 \\ -2x + 3y &\ge 3 \\ x \ge 0, y &\ge 0 \end{aligned}$$

22. Maximize $P = 2x + 5y$ subject to the constraints of Exercise 21.

23. Maximize $P = 3x + 4y$
subject to
$$\begin{aligned} x + 2y &\le 50 \\ 5x + 4y &\le 145 \\ 2x + y &\ge 25 \\ y \ge 5, x &\ge 0 \end{aligned}$$

24. Maximize $P = 4x - 3y$ subject to the constraints of Exercise 23.

25. Maximize $P = 2x + 3y$
subject to
$$\begin{aligned} x + y &\le 48 \\ x + 3y &\ge 60 \\ 9x + 5y &\le 320 \\ x \ge 10, y &\ge 0 \end{aligned}$$

26. Minimize $C = 5x + 3y$ subject to the constraints of Exercise 25.

27. Find the maximum and minimum of $P = 10x + 12y$ subject to

$$\begin{aligned} 5x + 2y &\geq 63 \\ x + y &\geq 18 \\ 3x + 2y &\leq 51 \\ x \geq 0, y &\geq 0 \end{aligned}$$

28. Find the maximum and minimum of $P = 4x + 3y$ subject to

$$\begin{aligned} 3x + 5y &\geq 20 \\ 3x + y &\leq 16 \\ -2x + y &\leq 1 \\ x \geq 0, y &\geq 0 \end{aligned}$$

The problems in Exercises 29–46 correspond to those in Exercises 1–18, Section 3.2. Use the results of your previous work to help you solve these problems.

29. Manufacturing—Production Scheduling A company manufactures two products, A and B, on two machines, I and II. It has been determined that the company will realize a profit of \$3/unit of product A and a profit of \$4/unit of product B. To manufacture a unit of product A requires 6 min on machine I and 5 min on machine II. To manufacture a unit of product B requires 9 min on machine I and 4 min on machine II. There are 5 hr of machine time available on machine I and 3 hr of machine time available on machine II in each work shift. How many units of each product should be produced in each shift to maximize the company's profit? What is the optimal profit?

30. Manufacturing—Production Scheduling National Business Machines manufactures two models of fax machines: A and B. Each model A costs \$100 to make, and each model B costs \$150. The profits are \$30 for each model A and \$40 for each model B fax machine. If the total number of fax machines demanded per month does not exceed 2500 and the company has earmarked no more than \$600,000/month for manufacturing costs, how many units of each model should National make each month in order to maximize its monthly profit? What is the optimal profit?

31. Manufacturing—Production Scheduling Kane Manufacturing has a division that produces two models of fireplace grates, model A and model B. To produce each model A grate requires 3 lb of cast iron and 6 min of labor. To produce each model B grate requires 4 lb of cast iron and 3 min of labor. The profit for each model A grate is \$2.00, and the profit for each model B grate is \$1.50. If 1000 lb of cast iron and 20 labor-hours are available for the production of fireplace grates per day, how many grates of each model should the division produce in order to maximize Kane's profit? What is the optimal profit?

32. Manufacturing—Production Scheduling Refer to Exercise 31. Because of a backlog of orders for model A grates, Kane's manager had decided to produce at least 150 of these models a day. Operating under this additional constraint, how many grates of each model should Kane produce to maximize profit? What is the optimal profit?

33. Manufacturing—Production Scheduling A division of the Winston Furniture Company manufactures dining tables and chairs. Each table requires 40 board feet of wood and 3 labor-hours. Each chair requires 16 board feet of wood and 4 labor-hours. The profit for each table is \$45, and the profit for each chair is \$20. In a certain week, the company has 3200 board feet of wood available and 520 labor-hours available. How many tables and chairs should Winston manufacture in order to maximize its profit? What is the maximum profit?

34. Manufacturing—Production Scheduling Refer to Exercise 33. If the profit for each table is \$50 and the profit for each chair is \$18, how many tables and chairs should Winston manufacture in order to maximize its profit? What is the maximum profit?

35. Finance—Allocation of Funds Madison Finance has a total of \$20 million earmarked for homeowner and auto loans. On the average, homeowner loans have a 10% annual rate of return, whereas auto loans yield a 12% annual rate of return. Management has also stipulated that the total amount of homeowner loans should be greater than or equal to 4 times the total amount of automobile loans. Determine the total amount of loans of each type that Madison should extend to each category in order to maximize its returns. What are the optimal returns?

36. Investments—Asset Allocation A financier plans to invest up to \$500,000 in two projects. Project A yields a return of 10% on the investment whereas project B yields a return of 15% on the investment. Because the investment in project B is riskier than the investment in project A, the financier has decided that the investment in project B should not exceed 40% of the total investment. How much should she invest in each project in order to maximize the return on her investment? What is the maximum return?

37. Manufacturing—Production Scheduling Acoustical manufactures a CD storage cabinet that can be bought fully assembled or as a kit. Each cabinet is processed in the fabrications department and the assembly department. If the fabrication department only manufactures fully assembled cabinets, then it can produce 200 units/day; and if it only manufactures kits, it can produce 200 units/day. If the assembly department produces only fully assembled cabinets, then it can produce 100 units/day; but if it produces only kits, then it can produce 300 units/day. Each fully assembled cabinet contributes \$50 to the profits of the company whereas each kit contributes \$40 to its profits. How many fully assembled units and how many kits should the company produce per day in order to maximize its profit? What is the optimal profit?

38. **AGRICULTURE—CROP PLANNING** A farmer plans to plant two crops, A and B. The cost of cultivating crop A is \$40/acre whereas that of crop B is \$60/acre. The farmer has a maximum of \$7400 available for land cultivation. Each acre of crop A requires 20 labor-hours, and each acre of crop B requires 25 labor-hours. The farmer has a maximum of 3300 labor-hours available. If she expects to make a profit of \$150/acre on crop A and \$200/acre on crop B, how many acres of each crop should she plant in order to maximize her profit? What is the optimal profit?

39. **MINING—PRODUCTION** Perth Mining Company operates two mines for the purpose of extracting gold and silver. The Saddle Mine costs \$14,000/day to operate, and it yields 50 oz of gold and 3000 oz of silver each day. The Horseshoe Mine costs \$16,000/day to operate, and it yields 75 oz of gold and 1000 oz of silver each day. Company management has set a target of at least 650 oz of gold and 18,000 oz of silver. How many days should each mine be operated so that the target can be met at a minimum cost? What is the minimum cost?

40. **TRANSPORTATION** Deluxe River Cruises operates a fleet of river vessels. The fleet has two types of vessels: A type-A vessel has 60 deluxe cabins and 160 standard cabins, whereas a type-B vessel has 80 deluxe cabins and 120 standard cabins. Under a charter agreement with Odyssey Travel Agency, Deluxe River Cruises is to provide Odyssey with a minimum of 360 deluxe and 680 standard cabins for their 15-day cruise in May. It costs \$44,000 to operate a type-A vessel and \$54,000 to operate a type-B vessel for that period. How many of each type vessel should be used in order to keep the operating costs to a minimum? What is the minimum cost?

41. **WATER SUPPLY** The water-supply manager for a Midwest city needs to supply the city with at least 10 million gal of potable (drinkable) water per day. The supply may be drawn from the local reservoir or from a pipeline to an adjacent town. The local reservoir has a maximum daily yield of 5 million gal of potable water, and the pipeline has a maximum daily yield of 10 million gallons. By contract, the pipeline is required to supply a minimum of 6 million gallons/day. If the cost for 1 million gallons of reservoir water is \$300 and that for pipeline water is \$500, how much water should the manager get from each source to minimize daily water costs for the city? What is the minimum daily cost?

42. **MANUFACTURING—PRODUCTION SCHEDULING** Ace Novelty manufactures "Giant Pandas" and "Saint Bernards." Each Panda requires 1.5 yd^2 of plush 30 ft^3 of stuffing, and 5 pieces of trim; each Saint Bernard requires 2 yd^2 of plush, 35 ft^3 of stuffing, and 8 pieces of trim. The profit for each Panda is \$10, and the profit for each Saint Bernard is \$15. If 3600 yd^2 of plush, 66,000 ft^3 of stuffing and 13,600 pieces of trim are available, how many of each of the stuffed animals should the company manufacture to maximize profit? What is the maximum profit?

43. **NUTRITION—DIET PLANNING** A nutritionist at the Medical Center has been asked to prepare a special diet for certain patients. She has decided that the meals should contain a minimum of 400 mg of calcium, 10 mg of iron, and 40 mg of vitamin C. She has further decided that the meals are to be prepared from foods A and B. Each ounce of food A contains 30 mg of calcium, 1 mg of iron, 2 mg of vitamin C, and 2 mg of cholesterol. Each ounce of food B contains 25 mg of calcium, 0.5 mg of iron, 5 mg of vitamin C, and 5 mg of cholesterol. Find how many ounces of each type of food should be used in a meal so that the cholesterol content is minimized and the minimum requirements of calcium, iron, and vitamin C are met.

44. **SOCIAL PROGRAMS PLANNING** AntiFam, a hunger-relief organization, has earmarked between \$2 and \$2.5 million (inclusive) for aid to two African countries, country A and country B. Country A is to receive between \$1 million and \$1.5 million (inclusive), and country B is to receive at least \$0.75 million. It has been estimated that each dollar spent in country A will yield an effective return of \$.60, whereas a dollar spent in country B will yield an effective return of \$.80. How should the aid be allocated if the money is to be utilized most effectively according to these criteria?
Hint: If x and y denote the amount of money to be given to country A and country B, respectively, then the objective function to be maximized is $P = 0.6x + 0.8y$.

45. **ADVERTISING** Everest Deluxe World Travel has decided to advertise in the Sunday editions of two major newspapers in town. These advertisements are directed at three groups of potential customers. Each advertisement in newspaper I is seen by 70,000 group-A customers, 40,000 group-B customers, and 20,000 group-C customers. Each advertisement in newspaper II is seen by 10,000 group-A, 20,000 group-B, and 40,000 group-C customers. Each advertisement in newspaper I costs \$1000, and each advertisement in newspaper II costs \$800. Everest would like their advertisements to be read by at least 2 million people from group A, 1.4 million people from group B, and 1 million people from group C. How many advertisements should Everest place in each newspaper to achieve its advertising goals at a minimum cost? What is the minimum cost?
Hint: Use different scales for drawing the feasible set.

46. **MANUFACTURING—SHIPPING COSTS** TMA manufactures 37-in. high definition LCD televisions in two separate locations, locations I and II. The output at location I is at most 6000 televisions/month, whereas the output at location II is at most 5000 televisions/month. TMA is the main supplier of televisions to the Pulsar Corporation, its holding company, which has priority in having all its requirements met. In a certain month, Pulsar placed orders for 3000 and 4000 televisions to be shipped to two of its factories located in city A and city B, respectively. The shipping costs (in dollars) per television from the two TMA plants to the two Pulsar factories are as follows:

	To Pulsar Factories	
From TMA	**City A**	**City B**
Location I	\$6	\$4
Location II	\$8	\$10

Find a shipping schedule that meets the requirements of both companies while keeping costs to a minimum.

47. Complete the solution to Example 3, Section 3.2.

48. Manufacturing—Production Scheduling Bata Aerobics manufactures two models of steppers used for aerobic exercises. Manufacturing each luxury model requires 10 lb of plastic and 10 min of labor. Manufacturing each standard model requires 16 lb of plastic and 8 min of labor. The profit for each luxury model is \$40, and the profit for each standard model is \$30. If 6000 lb of plastic and 60 labor-hours are available for the production of the steppers per day, how many steppers of each model should Bata produce each day in order to maximize its profit? What is the optimal profit?

49. Investment Planning Patricia has at most \$30,000 to invest in securities in the form of corporate stocks. She has narrowed her choices to two groups of stocks: growth stocks that she assumes will yield a 15% return (dividends and capital appreciation) within a year and speculative stocks that she assumes will yield a 25% return (mainly in capital appreciation) within a year. Determine how much she should invest in each group of stocks in order to maximize the return on her investments within a year if she has decided to invest at least 3 times as much in growth stocks as in speculative stocks.

50. Veterinary Science A veterinarian has been asked to prepare a diet for a group of dogs to be used in a nutrition study at the School of Animal Science. It has been stipulated that each serving should be no larger than 8 oz and must contain at least 29 units of nutrient I and 20 units of nutrient II. The vet has decided that the diet may be prepared from two brands of dog food: brand A and brand B. Each ounce of brand A contains 3 units of nutrient I and 4 units of nutrient II. Each ounce of brand B contains 5 units of nutrient I and 2 units of nutrient II. Brand A costs 3 cents/ounce and brand B costs 4 cents/ounce. Determine how many ounces of each brand of dog food should be used per serving to meet the given requirements at a minimum cost.

51. Market Research Trendex, a telephone survey company, has been hired to conduct a television-viewing poll among urban and suburban families in the Los Angeles area. The client has stipulated that a maximum of 1500 families is to be interviewed. At least 500 urban families must be interviewed, and at least half of the total number of families interviewed must be from the suburban area. For this service, Trendex will be paid \$6000 plus \$8 for each completed interview. From previous experience, Trendex has determined that it will incur an expense of \$4.40 for each successful interview with an urban family and \$5 for each successful interview with a suburban family. How many urban and suburban families should Trendex interview in order to maximize its profit?

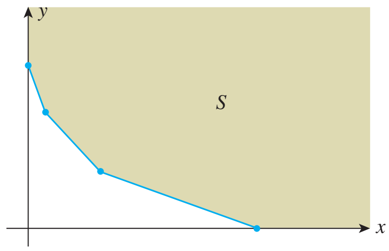
In Exercises 52–55, determine whether the statement is true or false. If it is true, explain why it is true. If it is false, give an example to show why it is false.

52. An optimal solution of a linear programming problem is a feasible solution, but a feasible solution of a linear programming problem need not be an optimal solution.

53. An optimal solution of a linear programming problem can occur inside the feasible set of the problem.

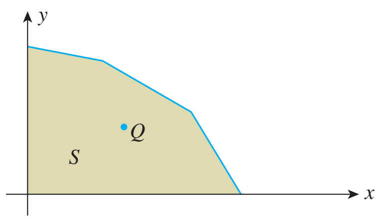
54. If a maximization problem has no solution, then the feasible set associated with the linear programming problem must be unbounded.

55. Suppose you are given the following linear programming problem: Maximize $P = ax + by$ on the unbounded feasible set S shown in the accompanying figure.



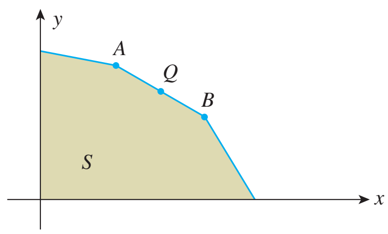
a. If $a > 0$ or $b > 0$, then the linear programming problem has no optimal solution.
b. If $a \leq 0$ and $b \leq 0$, then the linear programming problem has at least one optimal solution.

56. Suppose you are given the following linear programming problem: Maximize $P = ax + by$, where $a > 0$ and $b > 0$, on the feasible set S shown in the accompanying figure.



Explain, without using Theorem 1, why the optimal solution of the linear programming problem cannot occur at the point Q.

57. Suppose you are given the following linear programming problem: Maximize $P = ax + by$, where $a > 0$ and $b > 0$, on the feasible set S shown in the accompanying figure.



Explain, without using Theorem 1, why the optimal solution of the linear programming problem cannot occur at the point Q unless the problem has infinitely many solutions lying along the line segment joining the vertices A and B.
Hint: Let $A(x_1, y_1)$ and $B(x_2, y_2)$. Then $Q(\bar{x}, \bar{y})$, where $\bar{x} = x_1 + (x_2 - x_1)t$ and $\bar{y} = y_1 + (y_2 - y_1)t$ with $0 < t < 1$. Study the value of P at and near Q.

58. Consider the linear programming problem

$$\begin{aligned} \text{Maximize} \quad & P = 2x + 7y \\ \text{subject to} \quad & 2x + y \geq 8 \\ & x + y \geq 6 \\ & x \geq 0, y \geq 0 \end{aligned}$$

a. Sketch the feasible set S.
b. Find the corner points of S.
c. Find the values of P at the corner points of S found in part (b).
d. Show that the linear programming problem has no (optimal) solution. Does this contradict Theorem 1?

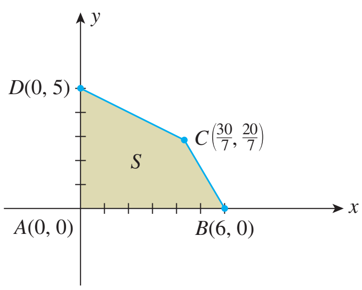
59. Consider the linear programming problem

$$\begin{aligned} \text{Minimize} \quad & C = -2x + 5y \\ \text{subject to} \quad & x + y \leq 3 \\ & 2x + y \leq 4 \\ & 5x + 8y \geq 40 \\ & x \geq 0, y \geq 0 \end{aligned}$$

a. Sketch the feasible set.
b. Find the solution(s) of the linear programming problem, if it exists.

3.3 Solutions to Self-Check Exercises

1. The feasible set S for the problem was graphed in the solution to Exercise 1, Self-Check Exercises 3.1. It is reproduced in the following figure.

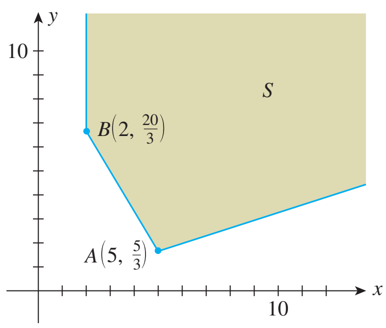


The values of the objective function P at the vertices of S are summarized in the following table.

Vertex	$P = 4x + 5y$
$A(0, 0)$	0
$B(6, 0)$	24
$C(\frac{30}{7}, \frac{20}{7})$	$\frac{220}{7} = 31\frac{3}{7}$
$D(0, 5)$	25

From the table, we see that the maximum for the objective function P is attained at the vertex $C(\frac{30}{7}, \frac{20}{7})$. Therefore, the solution to the problem is $x = \frac{30}{7}$, $y = \frac{20}{7}$, and $P = 31\frac{3}{7}$.

2. The feasible set S for the problem was graphed in the solution to Exercise 2, Self-Check Exercises 3.1. It is reproduced in the following figure.



Evaluating the objective function $C = 5x + 3y$ at each corner point, we obtain the table

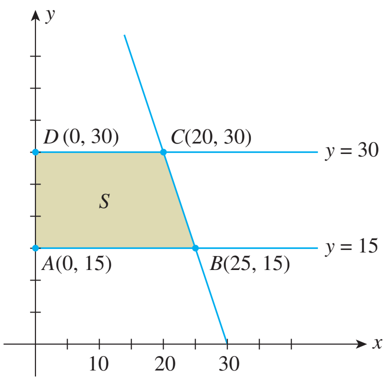
Vertex	$C = 5x + 3y$
$A(5, \frac{5}{3})$	30
$B(2, \frac{20}{3})$	30

We conclude that (i) the objective function is minimized at every point on the line segment joining the points $(5, \frac{5}{3})$ and $(2, \frac{20}{3})$, and (ii) the minimum value of C is 30.

3. Refer to Self-Check Exercise 3.2. The problem is to maximize $P = 50{,}000x + 20{,}000y$ subject to

$$\begin{aligned} 300x + 100y &\leq 9000 \\ y &\geq 15 \\ y &\leq 30 \\ x \geq 0, y &\geq 0 \end{aligned}$$

The feasible set S for the problem is shown in the following figure.



Evaluating the objective function $P = 50{,}000x + 20{,}000y$ at each vertex of S, we obtain

Vertex	$P = 50{,}000x + 20{,}000y$
$A(0, 15)$	300,000
$B(25, 15)$	1,550,000
$C(20, 30)$	1,600,000
$D(0, 30)$	600,000

From the table, we see that P is maximized when $x = 20$ and $y = 30$. Therefore, Gino should place 20 ads in the *City Tribune* and 30 in the *Daily News*.

3.4 Sensitivity Analysis

In this section, we investigate how changes in the parameters of a linear programming problem affect its optimal solution. This type of analysis is called **sensitivity analysis.** As in the previous sections, we restrict our analysis to the two-variable case, which is amenable to graphical analysis.

Recall the production problem posed in Example 1, Section 3.2, and solved in Example 1, Section 3.3:

$$\begin{array}{lrl} \text{Maximize} & P = x + 1.2y & \quad \text{Objective function} \\ \text{subject to} & 2x + y \le 180 & \quad \text{Constraint 1} \\ & x + 3y \le 300 & \quad \text{Constraint 2} \\ & x \ge 0, y \ge 0 & \end{array}$$

where x denotes the number of type-A souvenirs and y denotes the number of type-B souvenirs to be made. The optimal solution of this problem is $x = 48$, $y = 84$ (corresponding to the point C). The optimal value of P is 148.8 (Figure 17).

The following questions arise in connection with this production problem.

1. How do changes made to the coefficients of the objective function affect the optimal solution?
2. How do changes made to the constants on the right-hand side of the constraints affect the optimal solution?

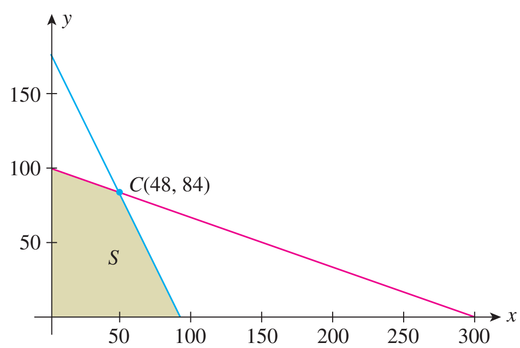


FIGURE 17
The optimal solution occurs at the point $C(48, 84)$.

Changes in the Coefficients of the Objective Function

In the production problem under consideration, the objective function is $P = x + 1.2y$. The coefficient of x, which is 1, tells us that the contribution to the profit for each type-A souvenir is \$1.00. The coefficient of y, 1.2, tells us that the contribution to the profit for each type-B souvenir is \$1.20. Now suppose the contribution to the profit for each type-B souvenir remains fixed at \$1.20 per souvenir. By how much can the contribution to the profit for each type-A souvenir vary without affecting the current optimal solution?

To answer this question, suppose the contribution to the profit of each type-A souvenir is \$$c$ so that

$$P = cx + 1.2y \tag{8}$$

We need to determine the range of values of c such that the solution remains optimal.

We begin by rewriting Equation (8) for the isoprofit line in the slope-intercept form. Thus,

$$y = -\frac{c}{1.2}x + \frac{P}{1.2} \tag{9}$$

The slope of the isoprofit line is $-c/1.2$. If the slope of the isoprofit line exceeds that of the line associated with constraint 2, then the optimal solution shifts from point C to point D (Figure 18).

On the other hand, if the slope of the isoprofit line is *less than or equal to* the slope of the line associated with constraint 2, then the optimal solution remains unaffected. (You may verify that $-\frac{1}{3}$ is the slope of the line associated with constraint 2 by writing the equation $x + 3y = 300$ in the slope-intercept form.) In other words, we must have

$$-\frac{c}{1.2} \le -\frac{1}{3}$$

$$\frac{c}{1.2} \ge \frac{1}{3} \qquad \text{Multiplying each side by } -1 \text{ reverses the inequality sign.}$$

$$c \ge \frac{1.2}{3} = 0.4$$

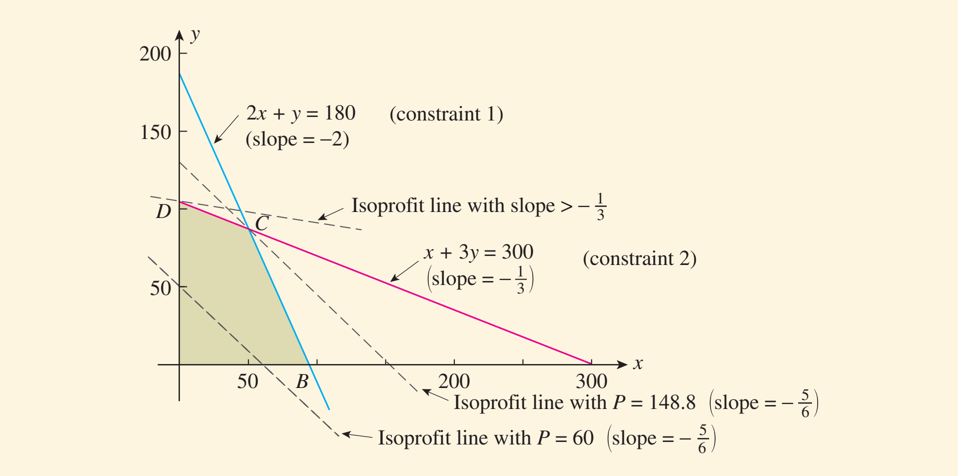


FIGURE 18
Increasing the slope of the isoprofit line $P = cx + 1.2y$ beyond $-\frac{1}{3}$ shifts the optimal solution from point C to point D.

A similar analysis shows that if the slope of the isoprofit line is less than that of the line associated with constraint 1, then the optimal solution shifts from point C to point B. Since the slope of the line associated with constraint 1 is -2, we see that point C will remain optimal provided that the slope of the isoprofit line is *greater than or equal to* -2; that is, if

$$-\frac{c}{1.2} \geq -2$$

$$\frac{c}{1.2} \leq 2$$

$$c \leq 2.4$$

Thus, we have shown that if $0.4 \leq c \leq 2.4$, then the optimal solution obtained previously remains unaffected.

This result tells us that if the contribution to the profit of each type-A souvenir lies between \$.40 and \$2.40, then Ace Novelty should still make 48 type-A souvenirs and 84 type-B souvenirs. Of course, the profit of the company will change with a change in the value of c—it's the product mix that stays the same. For example, if the contribution to the profit of a type-A souvenir is \$1.50, then the profit of the company will be \$172.80. (See Exercise 1.) Incidentally, our analysis shows that the parameter c is not a sensitive parameter.

We leave it as an exercise for you to show that, with the contribution to the profit of type-A souvenirs held constant at \$1.00 per souvenir, the contribution to each type-B souvenir can vary between \$.50 and \$3.00 without affecting the product mix for the optimal solution (see Exercise 1).

VIDEO

APPLIED EXAMPLE 1 Profit Function Analysis Kane Manufacturing has a division that produces two models of grates, model A and model B. To produce each model-A grate requires 3 pounds of cast iron and 6 minutes of labor. To produce each model-B grate requires 4 pounds of cast iron and 3 minutes of labor. The profit for each model A grate is \$2.00, and the profit for each model-B grate is \$1.50. Available for grate production each day are 1000 pounds of cast iron and 20 labor-hours. Because of an excess inventory of model-A grates, management has decided to limit the production of model-A grates to no more than 180 grates per day.

a. Use the method of corners to determine the number of grates of each model Kane should produce in order to maximize its profit.
b. Find the range of values that the contribution to the profit of a model-A grate can assume without changing the optimal solution.
c. Find the range of values that the contribution to the profit of a model-B grate can assume without changing the optimal solution.

Solution

a. Let x denote the number of model-A grates and y the number of model-B grates produced. Then verify that we are led to the following linear programming problem:

$$\begin{array}{lll} \text{Maximize} & P = 2x + 1.5y & \\ \text{subject to} & 3x + 4y \leq 1000 & \text{Constraint 1} \\ & 6x + 3y \leq 1200 & \text{Constraint 2} \\ & x \leq 180 & \text{Constraint 3} \\ & x \geq 0,\ y \geq 0 & \end{array}$$

The graph of the feasible set S is shown in Figure 19.

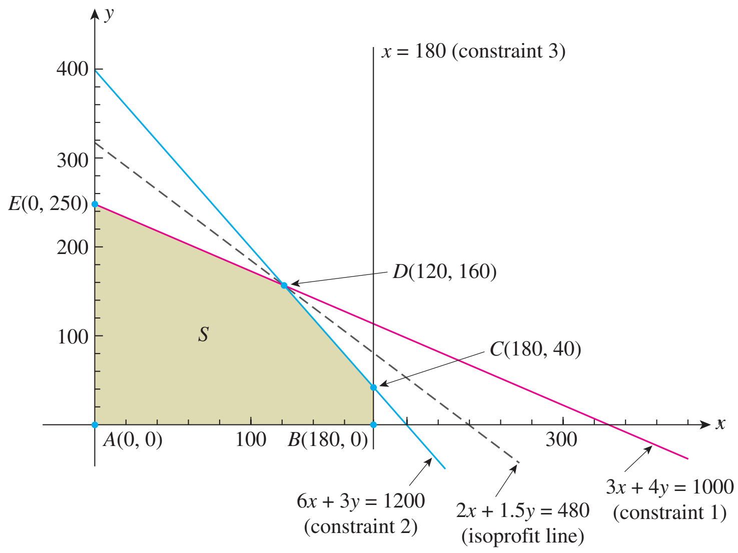


FIGURE 19
The shaded region is the feasible set S. Also shown are the lines of the equations associated with the constraints.

From the following table of values,

Vertex	$P = 2x + 1.5y$
$A(0, 0)$	0
$B(180, 0)$	360
$C(180, 40)$	420
$D(120, 160)$	480
$E(0, 250)$	375

we see that the maximum of $P = 2x + 1.5y$ occurs at the vertex $D(120, 160)$ with a value of 480. Thus, Kane realizes a maximum profit of \$480 per day by producing 120 model-A grates and 160 model-B grates each day.

b. Let c (in dollars) denote the contribution to the profit of a model-A grate. Then $P = cx + 1.5y$ or, upon solving for y,

$$\begin{aligned} y &= -\frac{c}{1.5}x + \frac{P}{1.5} \\ &= \left(-\frac{2}{3}c\right)x + \frac{2}{3}P \end{aligned}$$

Referring to Figure 19, you can see that if the slope of the isoprofit line is greater than the slope of the line associated with constraint 1, then the optimal solution will shift from point D to point E. Thus, for the optimal solution to remain unaffected, the slope of the isoprofit line must be less than or equal to the slope of the line associated with constraint 1. But the slope of the line associated with constraint 1 is $-\frac{3}{4}$, which you can see by rewriting the equation $3x + 4y = 1000$ in the slope-intercept form $y = -\frac{3}{4}x + 250$. Since the slope of the isoprofit line is $-2c/3$, we must have

$$\begin{aligned} -\frac{2c}{3} &\leq -\frac{3}{4} \\ \frac{2c}{3} &\geq \frac{3}{4} \\ c &\geq \left(\frac{3}{4}\right)\left(\frac{3}{2}\right) = \frac{9}{8} = 1.125 \end{aligned}$$

Again referring to Figure 19, you can see that if the slope of the isoprofit line is less than that of the line associated with constraint 2, then the optimal solution shifts from point D to point C. Since the slope of the line associated with constraint 2 is -2 (rewrite the equation $6x + 3y = 1200$ in the slope-intercept form $y = -2x + 400$), we see that the optimal solution remains at point D provided that the slope of the isoprofit line is greater than or equal to -2; that is,

$$-\frac{2c}{3} \geq -2$$

$$\frac{2c}{3} \leq 2$$

$$c \leq (2)\left(\frac{3}{2}\right) = 3$$

We conclude that the contribution to the profit of a model-A grate can assume values between \$1.125 and \$3.00 without changing the optimal solution.

c. Let c (in dollars) denote the contribution to the profit of a model-B grate. Then

$$P = 2x + cy$$

or, upon solving for y,

$$y = -\frac{2}{c}x + \frac{P}{c}$$

An analysis similar to that performed in part (b) with respect to constraint 1 shows that the optimal solution will remain in effect provided that

$$-\frac{2}{c} \leq -\frac{3}{4}$$

$$\frac{2}{c} \geq \frac{3}{4}$$

$$c \leq 2\left(\frac{4}{3}\right) = \frac{8}{3} = 2\frac{2}{3}$$

Performing an analysis with respect to constraint 2 shows that the optimal solution will remain in effect provided that

$$-\frac{2}{c} \geq -2$$

$$\frac{2}{c} \leq 2$$

$$c \geq 1$$

Thus, the contribution to the profit of a model-B grate can assume values between \$1.00 and \$2.67 without changing the optimal solution. ■

Changes to the Constants on the Right-Hand Side of the Constraint Inequalities

Let's return to the production problem posed at the beginning of this section:

$$\begin{array}{lll} \text{Maximize} & P = x + 1.2y & \\ \text{subject to} & 2x + y \leq 180 & \textit{Constraint 1} \\ & x + 3y \leq 300 & \textit{Constraint 2} \\ & x \geq 0, y \geq 0 & \end{array}$$

Now suppose the time available on machine I is changed from 180 minutes to $(180 + h)$ minutes, where h is a real number. Then the constraint on machine I is changed to

$$2x + y \leq 180 + h$$

Observe that the line with equation $2x + y = 180 + h$ is parallel to the line $2x + y = 180$ associated with the original constraint 1.

As you can see from Figure 20, the result of adding the constant h to the right-hand side of constraint 1 is to shift the current optimal solution from the point C to the new optimal solution occurring at the point C'. To find the coordinates of C', we observe that C' is the point of intersection of the lines with equations

$$2x + y = 180 + h \quad \text{and} \quad x + 3y = 300$$

Thus, the coordinates of the point are found by solving the system of linear equations

$$\begin{aligned} 2x + y &= 180 + h \\ x + 3y &= 300 \end{aligned}$$

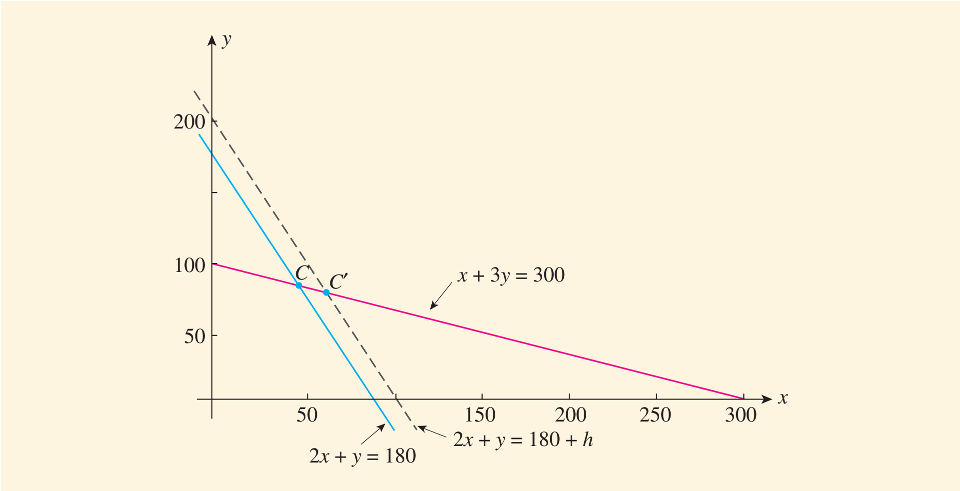


FIGURE 20
The lines with equations $2x + y = 180$ and $2x + y = 180 + h$ are parallel to each other.

The solutions are

$$x = \frac{3}{5}(80 + h) \quad \text{and} \quad y = \frac{1}{5}(420 - h) \tag{10}$$

The nonnegativity of x implies that

$$\begin{aligned} \frac{3}{5}(80 + h) &\geq 0 \\ 80 + h &\geq 0 \\ h &\geq -80 \end{aligned}$$

Next, the nonnegativity of y implies that

$$\begin{aligned} \frac{1}{5}(420 - h) &\geq 0 \\ 420 - h &\geq 0 \\ h &\leq 420 \end{aligned}$$

Thus, h must satisfy the inequalities $-80 \leq h \leq 420$. Our computations reveal that a meaningful solution will require that the time available for machine I must range between $(180 - 80)$ and $(180 + 420)$ minutes—that is, between 100 and 600 minutes.

Under these conditions, Ace Novelty should produce $\frac{3}{5}(80 + h)$ type-A souvenirs and $\frac{1}{5}(420 - h)$ type-B souvenirs.

For example, if Ace Novelty can manage to increase the time available on machine I by 10 minutes, then it should produce $\frac{3}{5}(80 + 10)$, or 54, type-A souvenirs and $\frac{1}{5}(420 - 10)$, or 82, type-B souvenirs; the resulting profit is

$$P = x + 1.2y = 54 + (1.2)(82) = 152.4$$

or \$152.40.

We leave it as an exercise for you to show that if the time available on machine II is changed from 300 minutes to $(300 + k)$ minutes with no change in the maximum capacity for machine I, then k must satisfy the inequalities $-210 \leq k \leq 240$. Thus, for a meaningful solution to the problem, the time available on machine II must lie between 90 and 540 min (see Exercise 2). Furthermore, in this case, Ace Novelty should produce $\frac{1}{5}(240 - k)$ type-A souvenirs and $\frac{1}{5}(420 + 2k)$ type-B souvenirs (see Exercise 2).

Shadow Prices

We have just seen that if Ace Novelty could increase the maximum available time on machine I by 10 minutes, then the profit would increase from the original optimal value of \$148.80 to \$152.40. In this case, finding the extra time on machine I proved beneficial to the company. More generally, to study the economic benefits that can be derived from increasing its resources, a company looks at the shadow prices associated with the respective resources. We define the **shadow price** for the ith resource (associated with the ith constraint of the linear programming problem) to be the amount by which the value of the objective function is improved—increased in a maximization problem and decreased in a minimization problem—if the right-hand side of the ith constraint is changed by 1 unit.

In the Ace Novelty example discussed earlier, we showed that if the right-hand side of constraint 1 is increased by h units then the optimal solution is given by (10):

$$x = \frac{3}{5}(80 + h) \quad \text{and} \quad y = \frac{1}{5}(420 - h)$$

The resulting profit is calculated as follows:

$$\begin{aligned} P &= x + 1.2y \\ &= x + \frac{6}{5}y \\ &= \frac{3}{5}(80 + h) + \left(\frac{6}{5}\right)\left(\frac{1}{5}\right)(420 - h) \\ &= \frac{3}{25}(1240 + 3h) \end{aligned}$$

Upon setting $h = 1$, we find

$$\begin{aligned} P &= \frac{3}{25}(1240 + 3) \\ &= 149.16 \end{aligned}$$

Since the optimal profit for the original problem is \$148.80, we see that the shadow price for the first resource is $149.16 - 148.80$, or \$.36. To summarize, Ace Novelty's profit increases at the rate of \$.36 per 1-minute increase in the time available on machine I.

We leave it as an exercise for you to show that the shadow price for resource 2 (associated with constraint 2) is \$.28 (see Exercise 2).

PORTFOLIO Morgan Wilson

TITLE Land Use Planner
INSTITUTION City of Burien

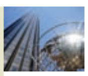
As a land use planner for the city of Burien, Washington, I assist property owners every day in the development of their land. By definition, land use planners develop plans and recommend policies for managing land use. To do this, I must take into account many existing and potential factors such as public transportation, zoning laws, and other municipal laws. By using the basic ideas of linear programming, I work with the property owners to figure out maximum and minimum use requirements for each individual situation. Then, I am able to review and evaluate proposals for land use plans and prepare recommendations. All this is necessary to process an application for a land development permit.

Here's how it works. A property owner who wants to start a business on a vacant commercially zoned piece of property comes to me. We have a discussion to find out what type of commercial zone the property is in and whether the use is permitted or would require additional land use review. If the use is permitted and no further land use review is required, I let the applicant know what criteria have to be met and shown on the building plans. At this point, the applicant begins working with his or her building contractor, architect, or engineer and landscape architect to meet the zoning code criteria. Once the applicant has worked with one or more of these professionals, building plans can be submitted for review. The plans are routed to several different departments (building, engineer, public works, and the fire department). Because I am the land use planner for the project, one set of plans is routed to my desk for review.

During this review, I determine whether or not the zoning requirements have been met in order to make a final determination of the application. These zoning requirements are assessed by asking the applicant to give us a site plan showing lot area measurements, building and impervious surface coverage calculations, and building setbacks, just to name a few. Additionally, I have to determine the parking requirements. How many off-street parking spaces are required? What are the isle widths? Is there enough room for backing up? Then, I look at the landscaping requirements. Plans need to be drawn up by a landscape architect and list specifics about the location, size, and types of plants that will be used.

By weighing all these factors and measurements, I can determine the viability of a land development project. The basic ideas of linear programming are fundamentally at the heart of this determination and are key to the day-to-day choices that I must make in my profession.

© Martin Shields/Alamy



APPLIED EXAMPLE 2 Shadow Prices Consider the problem posed in Example 1:

$$
\begin{aligned}
\text{Maximize} \quad & P = 2x + 1.5y \\
\text{subject to} \quad & 3x + 4y \le 1000 && \text{Constraint 1} \\
& 6x + 3y \le 1200 && \text{Constraint 2} \\
& x \le 180 && \text{Constraint 3} \\
& x \ge 0,\ y \ge 0
\end{aligned}
$$

a. Find the range of values that resource 1 (the constant on the right-hand side of constraint 1) can assume.

b. Find the shadow price for resource 1.

Solution

a. Suppose the right-hand side of constraint 1 is replaced by $1000 + h$, where h is a real number. Then the new optimal solution occurs at the point D' (Figure 21).

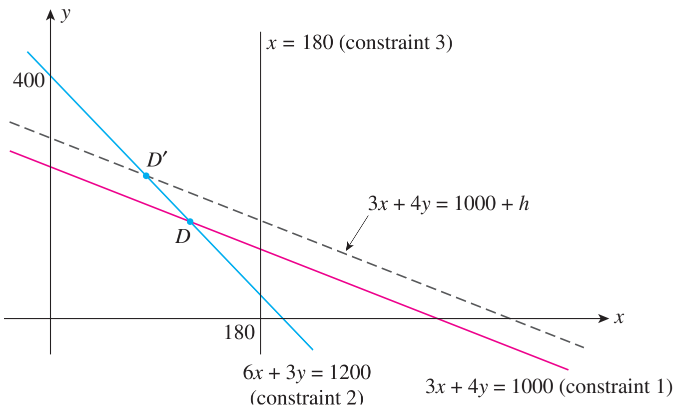


FIGURE 21
As the amount of resource 1 changes, the point at which the optimal solution occurs shifts from *D* to *D*′.

To find the coordinates of D', we solve the system

$$3x + 4y = 1000 + h$$
$$6x + 3y = 1200$$

Multiplying the first equation by -2 and then adding the resulting equation to the second equation gives

$$-5y = -800 - 2h$$
$$y = \frac{2}{5}(400 + h)$$

$$\begin{aligned} -6x - 8y &= -2000 - 2h \\ 6x + 3y &= 1200 \\ \hline -5y &= -800 - 2h \end{aligned}$$

Substituting this value of y into the second equation in the system gives

$$6x + \frac{6}{5}(400 + h) = 1200$$
$$x + \frac{1}{5}(400 + h) = 200$$
$$x = \frac{1}{5}(600 - h)$$

The nonnegativity of y implies that $h \geq -400$, and the nonnegativity of x implies that $h \leq 600$. But constraint 3 dictates that x must also satisfy

$$x = \frac{1}{5}(600 - h) \leq 180$$
$$600 - h \leq 900$$
$$-h \leq 300$$
$$h \geq -300$$

Therefore, h must satisfy $-300 \leq h \leq 600$. This tells us that the amount of resource 1 must lie between $1000 - 300$, or 700, and $1000 + 600$, or 1600—that is, between 700 and 1600 pounds.

b. If we set $h = 1$ in part (a), we obtain

$$x = \frac{1}{5}(600 - 1) = \frac{599}{5}$$
$$y = \frac{2}{5}(400 + 1) = \frac{802}{5}$$

Therefore, the profit realized at this level of production is

$$\begin{aligned} P &= 2x + \frac{3}{2}y = 2\left(\frac{599}{5}\right) + \frac{3}{2}\left(\frac{802}{5}\right) \\ &= \frac{2401}{5} = 480.2 \end{aligned}$$

Since the original optimal profit is \$480 (see Example 1), we see that the shadow price for resource 1 is \$.20. ■

If you examine Figure 21, you can see that increasing resource 3 (the constant on the right-hand side of constraint 3) has no effect on the optimal solution $D(120, 160)$ of the problem at hand. In other words, an increase in the resource associated with constraint 3 has no economic benefit for Kane Manufacturing. The shadow price for this resource is *zero*. There is a *surplus* of this resource. Hence, we say that the constraint $x \leq 180$ is not binding on the optimal solution $D(120, 160)$.

On the other hand, constraints 1 and 2, which *hold with equality* at the optimal solution $D(120, 160)$, are said to be **binding constraints.** The objective function cannot be increased without increasing these resources. They have *positive* shadow prices.

Importance of Sensitivity Analysis

We conclude this section by pointing out the importance of sensitivity analysis in solving real-world problems. The values of the parameters in these problems may change. For example, the management of Ace Novelty might wish to increase the price of a type-A souvenir because of increased demand for the product, or they might want to see how a change in the time available on machine I affects the (optimal) profit of the company.

When a parameter of a linear programming problem is changed, it is true that one need only re-solve the problem to obtain a new solution to the problem. But since a real-world linear programming problem often involves thousands of parameters, the amount of work involved in finding a new solution is prohibitive. Another disadvantage in using this approach is that it often takes many trials with different values of a parameter in order to see their effect on the optimal solution of the problem. Thus, a more analytical approach such as that discussed in this section is desirable.

Returning to the discussion of Ace Novelty, our analysis of the changes in the coefficients of the objective (profit) function suggests that if management decides to raise the price of a type-A souvenir, it can do so with the assurance that the optimal solution holds as long as the new price leaves the contribution to the profit of a type-A souvenir between \$.40 and \$2.40. There is no need to re-solve the linear programming problem for each new price being considered. Also, our analysis of the changes in the parameters on the right-hand side of the constraints suggests, for example, that a meaningful solution to the problem requires that the time available for machine I lie in the range between 100 and 600 minutes. Furthermore, the analysis tells us how to compute the increase (decrease) in the optimal profit when the resource is adjusted, by using the shadow price associated with that constraint. Again, there is no need to re-solve the linear programming problem each time a change in the resource available is anticipated.

Using Technology examples and exercises that are solved using Excel's Solver can be found on pages 222–226 and 237–241.

3.4 Self-Check Exercises

Consider the linear programming problem:

$$\begin{array}{lll} \text{Maximize} & P = 2x + 4y & \\ \text{subject to} & 2x + 5y \leq 19 & \text{Constraint 1} \\ & 3x + 2y \leq 12 & \text{Constraint 2} \\ & x \geq 0, y \geq 0 & \end{array}$$

1. Use the method of corners to solve this problem.
2. Find the range of values that the coefficient of x can assume without changing the optimal solution.
3. Find the range of values that resource 1 (the constant on the right-hand side of constraint 1) can assume without changing the optimal solution.
4. Find the shadow price for resource 1.
5. Identify the binding and nonbinding constraints.

Solutions to Self-Check Exercises can be found on page 197.

3.4 Concept Questions

1. Suppose $P = 3x + 4y$ is the objective function in a linear programming (maximization) problem, where x denotes the number of units of product A and y denotes the number of units of product B to be made. What does the coefficient of x represent? The coefficient of y?
2. Given the linear programming problem

$$\begin{array}{lll} \text{Maximize} & P = 3x + 4y & \\ \text{subject to} & x + y \leq 4 & \text{Resource 1} \\ & 2x + y \leq 5 & \text{Resource 2} \end{array}$$

   a. Write the inequality that represents an increase of h units in resource 1.
   b. Write the inequality that represents an increase of k units in resource 2.
3. Explain the meaning of (a) a shadow price and (b) a binding constraint.

3.4 Exercises

1. Refer to the production problem discussed on pages 185–187.
   a. Show that the optimal solution holds if the contribution to the profit of a type-B souvenir lies between \$.50 and \$3.00.
   b. Show that if the contribution to the profit of a type-A souvenir is \$1.50 (with the contribution to the profit of a type-B souvenir held at \$1.20), then the optimal profit of the company will be \$172.80.
   c. What will be the optimal profit of the company if the contribution to the profit of a type-B souvenir is \$2.00 (with the contribution to the profit of a type-A souvenir held at \$1.00)?
2. Refer to the production problem discussed on pages 189–191.
   a. Show that, for a meaningful solution, the time available on machine II must lie between 90 and 540 min.
   b. Show that, if the time available on machine II is changed from 300 min to $(300 + k)$ min, with no change in the maximum capacity for machine I, then Ace Novelty's profit is maximized by producing $\frac{1}{5}(240 - k)$ type-A souvenirs and $\frac{1}{5}(420 + 2k)$ type-B souvenirs, where $-210 \leq k \leq 240$.
   c. Show that the shadow price for resource 2 (associated with constraint 2) is \$.28.
3. Refer to Example 2.
   a. Find the range of values that resource 2 can assume.
   b. By how much can the right-hand side of constraint 3 be changed such that the current optimal solution still holds?
4. Refer to Example 2.
   a. Find the shadow price for resource 2.
   b. Identify the binding and nonbinding constraints.

In Exercises 5–10, you are given a linear programming problem.
a. Use the method of corners to solve the problem.
b. Find the range of values that the coefficient of x can assume without changing the optimal solution.
c. Find the range of values that resource 1 (requirement 1) can assume.
d. Find the shadow price for resource 1 (requirement 1).
e. Identify the binding and nonbinding constraints.

5. $$\begin{array}{lll} \text{Maximize} & P = 3x + 4y & \\ \text{subject to} & 2x + 3y \leq 12 & \text{Resource 1} \\ & 2x + y \leq 8 & \text{Resource 2} \\ & x \geq 0, y \geq 0 & \end{array}$$

6. Maximize $P = 2x + 5y$
subject to
$x + 3y \le 15$ Resource 1
$4x + y \le 16$ Resource 2
$x \ge 0, y \ge 0$

7. Minimize $C = 2x + 5y$
subject to
$x + 2y \ge 4$ Requirement 1
$x + y \ge 3$ Requirement 2
$x \ge 0, y \ge 0$

8. Minimize $C = 3x + 4y$
subject to
$x + 3y \ge 8$ Requirement 1
$x + y \ge 4$ Requirement 2
$x \ge 0, y \ge 0$

9. Maximize $P = 4x + 3y$
subject to
$5x + 3y \le 30$ Resource 1
$2x + 3y \le 21$ Resource 2
$x \le 4$ Resource 3
$x \ge 0, y \ge 0$

10. Maximize $P = 4x + 5y$
subject to
$x + y \le 30$ Resource 1
$x + 2y \le 40$ Resource 2
$x \le 25$ Resource 3
$x \ge 0, y \ge 0$

11. **MANUFACTURING—PRODUCTION SCHEDULING** A company manufactures two products, A and B, on machines I and II. The company will realize a profit of \$3/unit of product A and a profit of \$4/unit of product B. Manufacturing 1 unit of product A requires 6 min on machine I and 5 min on machine II. Manufacturing 1 unit of product B requires 9 min on machine I and 4 min on machine II. There are 5 hr of time available on machine I and 3 hr of time available on machine II in each work shift.

a. How many units of each product should be produced in each shift to maximize the company's profit?
b. Find the range of values that the contribution to the profit of 1 unit of product A can assume without changing the optimal solution.
c. Find the range of values that the resource associated with the time constraint on machine I can assume.
d. Find the shadow price for the resource associated with the time constraint on machine I.

12. **AGRICULTURE—CROP PLANNING** A farmer plans to plant two crops, A and B. The cost of cultivating crop A is \$40/acre whereas that of crop B is \$60/acre. The farmer has a maximum of \$7400 available for land cultivation. Each acre of crop A requires 20 labor-hours, and each acre of crop B requires 25 labor-hours. The farmer has a maximum of 3300 labor-hours available. If he expects to make a profit of \$150/acre on crop A and \$200/acre on crop B, how many acres of each crop should he plant in order to maximize his profit?

a. Find the range of values that the contribution to the profit of an acre of crop A can assume without changing the optimal solution.
b. Find the range of values that the resource associated with the constraint on the available land can assume.
c. Find the shadow price for the resource associated with the constraint on the available land.

13. **MINING—PRODUCTION** Perth Mining Company operates two mines for the purpose of extracting gold and silver. The Saddle Mine costs \$14,000/day to operate, and it yields 50 oz of gold and 3000 oz of silver per day. The Horseshoe Mine costs \$16,000/day to operate, and it yields 75 oz of gold and 1000 ounces of silver per day. Company management has set a target of at least 650 oz of gold and 18,000 oz of silver.

a. How many days should each mine be operated so that the target can be met at a minimum cost?
b. Find the range of values that the Saddle Mine's daily operating cost can assume without changing the optimal solution.
c. Find the range of values that the requirement for gold can assume.
d. Find the shadow price for the requirement for gold.

14. **TRANSPORTATION** Deluxe River Cruises operates a fleet of river vessels. The fleet has two types of vessels: a type-A vessel has 60 deluxe cabins and 160 standard cabins, whereas a type-B vessel has 80 deluxe cabins and 120 standard cabins. Under a charter agreement with the Odyssey Travel Agency, Deluxe River Cruises is to provide Odyssey with a minimum of 360 deluxe and 680 standard cabins for their 15-day cruise in May. It costs \$44,000 to operate a type-A vessel and \$54,000 to operate a type-B vessel for that period.

a. How many of each type of vessel should be used in order to keep the operating costs to a minimum?
b. Find the range of values that the cost of operating a type-A vessel can assume without changing the optimal solution.
c. Find the range of values that the requirement for deluxe cabins can assume.
d. Find the shadow price for the requirement for deluxe cabins.

15. **MANUFACTURING—PRODUCTION SCHEDULING** Soundex produces two models of satellite radios. Model A requires 15 min of work on assembly line I and 10 min of work on assembly line II. Model B requires 10 min of work on assembly line I and 12 min of work on assembly line II. At most 25 hr of assembly time on line I and 22 hr of assembly time on line II are available each day. Soundex anticipates a profit of \$12 on model A and \$10 on model B. Because of previous overproduction, management decides to limit the production of model A satellite radios to no more than 80/day.

a. To maximize Soundex's profit, how many satellite radios of each model should be produced each day?
b. Find the range of values that the contribution to the profit of a model A satellite radio can assume without changing the optimal solution.
c. Find the range of values that the resource associated with the time constraint on machine I can assume.
d. Find the shadow price for the resource associated with the time constraint on machine I.
e. Identify the binding and nonbinding constraints.

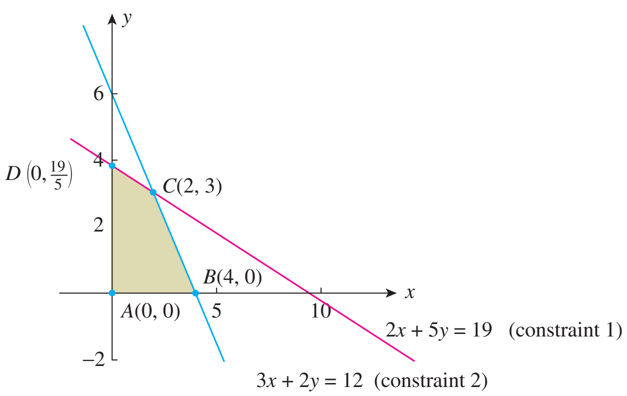
16. **Manufacturing** Refer to Exercise 15.
 a. If the contribution to the profit of a model A satellite radio is changed to \$8.50/radio, will the original optimal solution still hold? What will be the optimal profit?
 b. If the contribution to the profit of a model A satellite radio is changed to \$14.00/radio, will the original optimal solution still hold? What will be the optimal profit?

17. **Manufacturing—Production Scheduling** Kane Manufacturing has a division that produces two models of fireplace grates, model A and model B. To produce each model-A grate requires 3 lb of cast iron and 6 min of labor. To produce each model-B grate requires 4 lb of cast iron and 3 min of labor. The profit for each model-A grate is \$2, and the profit for each model-B grate is \$1.50. 1000 lb of cast iron and 20 labor-hours are available for the production of grates each day. Because of an excess inventory of model-B grates, management has decided to limit the production of model-B grates to no more than 200 grates per day. How many grates of each model should the division produce daily to maximize Kane's profit?
 a. Use the method of corners to solve the problem.
 b. Find the range of values that the contribution to the profit of a model-A grate can assume without changing the optimal solution.
 c. Find the range of values that the resource for cast iron can assume without changing the optimal solution.
 d. Find the shadow price for the resource for cast iron.
 e. Identify the binding and nonbinding constraints.

18. **Manufacturing** Refer to Exercise 17.
 a. If the contribution to the profit of a model-A grate is changed to \$1.75/grate, will the original optimal solution still hold? What will be the new optimal solution?
 b. If the contribution to the profit of a model-A grate is changed to \$2.50/grate, will the original optimal solution still hold? What will be the new optimal solution?

3.4 Solutions to Self-Check Exercises

1. The feasible set for the problem is shown in the accompanying figure.



Evaluating the objective function $P = 2x + 4y$ at each feasible corner point, we obtain the following table:

Vertex	$P = 2x + 4y$
$A(0, 0)$	0
$B(4, 0)$	8
$C(2, 3)$	16
$D(0, \frac{19}{5})$	$15\frac{1}{5}$

We conclude that the maximum value of P is 16 attained at the point (2, 3).

2. Assume that $P = cx + 4y$. Then

$$y = -\frac{c}{4}x + \frac{P}{4}$$

The slope of the isoprofit line is $-\frac{c}{4}$ and must be less than or equal to the slope of the line associated with constraint 1; that is,

$$-\frac{c}{4} \leq -\frac{2}{5}$$

Solving, we find $c \geq \frac{8}{5}$. A similar analysis shows that the slope of the isoprofit line must be greater than or equal to the slope of the line associated with constraint 2; that is,

$$-\frac{c}{4} \geq -\frac{3}{2}$$

Solving, we find $c \leq 6$. Thus, we have shown that if $1.6 \leq c \leq 6$ then the optimal solution obtained previously remains unaffected.

3. Suppose the right-hand side of constraint 1 is replaced by $19 + h$, where h is a real number. Then the new optimal solution occurs at the point whose coordinates are found by solving the system

$$2x + 5y = 19 + h$$
$$3x + 2y = 12$$

Multiplying the second equation by -5 and adding the resulting equation to 2 times the first equation, we obtain

$$-11x = -60 + 2(19 + h) = -22 + 2h$$
$$x = 2 - \frac{2}{11}h$$

Substituting this value of x into the second equation in the system gives

$$3\left(2 - \frac{2}{11}h\right) + 2y = 12$$
$$2y = 12 - 6 + \frac{6}{11}h$$
$$y = 3 + \frac{3}{11}h$$

The nonnegativity of x implies $2 - \frac{2}{11}h \geq 0$, or $h \leq 11$. The nonnegativity of y implies $3 + \frac{3}{11}h \geq 0$, or $h \geq -11$. Therefore, h must satisfy $-11 \leq h \leq 11$. This tells us that the amount used of resource 1 must lie between $19 - 11$ and $19 + 11$—that is, between 8 and 30.

4. If we set $h = 1$ in Exercise 3, we find that $x = \frac{20}{11}$ and $y = \frac{36}{11}$. Therefore, for these values of x and y,

$$P = 2\left(\frac{20}{11}\right) + 4\left(\frac{36}{11}\right) = \frac{184}{11} = 16\frac{8}{11}$$

Since the original optimal value of P is 16, we see that the shadow price for resource 1 is $\frac{8}{11}$.

5. Since both constraints hold with equality at the optimal solution $C(2, 3)$, they are binding constraints.

CHAPTER 3 Summary of Principal Terms

TERMS

solution set of a system of linear inequalities (159)
bounded solution set (160)
unbounded solution set (160)
objective function (164)
linear programming problem (164)
feasible solution (172)
feasible set (172)
optimal solution (172)
isoprofit line (173)
method of corners (174)
sensitivity analysis (185)
shadow price (191)
binding constraint (194)

CHAPTER 3 Concept Review Questions

Fill in the blanks.

1. **a.** The solution set of the inequality $ax + by < c$ (a, b not both zero) is a/an _____ _____ that does not include the _____ with equation $ax + by = c$.
   b. If $ax + by < c$ describes the lower half-plane, then the inequality _____ describes the lower half-plane together with the line having equation _____.

2. **a.** The solution set of a system of linear inequalities in the two variables x and y is the set of all _____ satisfying _____ inequality of the system.
   b. The solution set of a system of linear inequalities is _____ if it can be _____ by a circle.

3. A linear programming problem consists of a linear function, called a/an _____ _____ to be _____ or _____ subject to constraints in the form of _____ equations or _____.

4. **a.** If a linear programming problem has a solution, then it must occur at a/an _____ _____ of the feasible set.
   b. If the objective function of a linear programming problem is optimized at two adjacent vertices of the feasible set, then it is optimized at every point on the _____ segment joining these vertices.

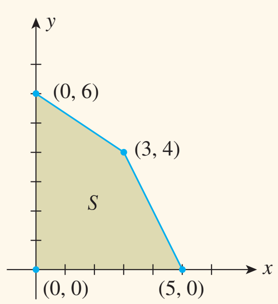
5. In sensitivity analysis, we investigate how changes in the _____ of a linear programming problem affect the _____ solution.

6. The shadow price for the ith _____ is the _____ by which the _____ of the objective function is _____ if the right-hand side of the ith constraint is _____ by 1 unit.

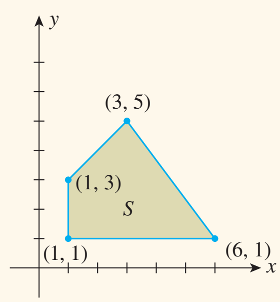
CHAPTER 3 Review Exercises

In Exercises 1 and 2, find the optimal value(s) of the objective function on the feasible set *S*.

1. $Z = 2x + 3y$



2. $Z = 4x + 3y$



In Exercises 3–12, use the method of corners to solve the linear programming problem.

3. Maximize $P = 3x + 5y$
subject to
$$\begin{aligned} 2x + 3y &\le 12 \\ x + y &\le 5 \\ x \ge 0, y &\ge 0 \end{aligned}$$

4. Maximize $P = 2x + 3y$
subject to
$$\begin{aligned} 2x + y &\le 12 \\ x - 2y &\le 1 \\ x \ge 0, y &\ge 0 \end{aligned}$$

5. Minimize $C = 2x + 5y$
subject to
$$\begin{aligned} x + 3y &\ge 15 \\ 4x + y &\ge 16 \\ x \ge 0, y &\ge 0 \end{aligned}$$

6. Minimize $C = 3x + 4y$
subject to
$$\begin{aligned} 2x + y &\ge 4 \\ 2x + 5y &\ge 10 \\ x \ge 0, y &\ge 0 \end{aligned}$$

7. Maximize $P = 3x + 2y$
subject to
$$\begin{aligned} 2x + y &\le 16 \\ 2x + 3y &\le 36 \\ 4x + 5y &\ge 28 \\ x \ge 0, y &\ge 0 \end{aligned}$$

8. Maximize $P = 6x + 2y$
subject to
$$\begin{aligned} x + 2y &\le 12 \\ x + y &\le 8 \\ 2x - 3y &\ge 6 \\ x \ge 0, y &\ge 0 \end{aligned}$$

9. Minimize $C = 2x + 7y$
subject to
$$\begin{aligned} 3x + 5y &\ge 45 \\ 3x + 10y &\ge 60 \\ x \ge 0, y &\ge 0 \end{aligned}$$

10. Minimize $C = 3x + 2y$
subject to
$$\begin{aligned} 2x + y &\ge 8 \\ x + y &\ge 6 \\ 2x + 3y &\ge 14 \\ x \ge 0, y &\ge 0 \end{aligned}$$

11. Minimize $C = 4x + y$
subject to
$$\begin{aligned} 6x + y &\ge 18 \\ 2x + y &\ge 10 \\ x + 4y &\ge 12 \\ x \ge 0, y &\ge 0 \end{aligned}$$

12. Find the maximum and minimum values of $Q = 3x + 4y$ subject to
$$\begin{aligned} x - y &\ge -10 \\ x + 3y &\ge 30 \\ 7x + 4y &\le 140 \end{aligned}$$

13. Find the maximum and minimum of $Q = x + y$ subject to
$$\begin{aligned} 5x + 2y &\ge 20 \\ x + 2y &\ge 8 \\ x + 4y &\le 22 \\ x \ge 0, y &\ge 0 \end{aligned}$$

14. Find the maximum and minimum of $Q = 2x + 5y$ subject to
$$\begin{aligned} x + y &\ge 4 \\ -x + y &\le 6 \\ x + 3y &\le 30 \\ x &\le 12 \\ x \ge 0, y &\ge 0 \end{aligned}$$

15. **Financial Analysis** An investor has decided to commit no more than \$80,000 to the purchase of the common stocks of two companies, company A and company B. He has also estimated that there is a chance of at most a 1% capital loss on his investment in company A and a chance of at most a 4% loss on his investment in company B, and he has decided that together these losses should not exceed \$2000. On the other hand, he expects to make a 14% profit from his investment in company A and a 20% profit from his investment in company B. Determine how much he should invest in the stock of each company in order to maximize his investment returns.

16. **Manufacturing—Production Scheduling** Soundex produces two models of satellite radios. Model A requires 15 min of work on assembly line I and 10 min of work on assembly line II. Model B requires 10 min of work on assembly line I and 12 min of work on assembly line II. At most, 25 labor-hours of assembly time on line I and 22 labor-hours of assembly time on line II are available each day. It is anticipated that Soundex will realize a profit of \$12 on model A and \$10 on model B. How many satellite radios of each model should be produced each day in order to maximize Soundex's profit?

17. **Manufacturing—Production Scheduling** Kane Manufacturing has a division that produces two models of grates, model A and model B. To produce each model-A grate requires 3 lb of cast iron and 6 min of labor. To produce each model-B grate requires 4 lb of cast iron and 3 min of labor. The profit for each model-A grate is \$2.00, and the profit for each model-B grate is \$1.50. Available for grate production each day are 1000 lb of cast iron and 20 labor-hours. Because of a backlog of orders for model-B grates, Kane's manager has decided to produce at least 180 model-B grates/day. How many grates of each model should Kane produce to maximize its profit?

18. **Minimizing Shipping Costs** A manufacturer of projection TVs must ship a total of at least 1000 TVs to its two central warehouses. Each warehouse can hold a maximum of 750 TVs. The first warehouse already has 150 TVs on hand, whereas the second has 50 TVs on hand. It costs \$8 to ship a TV to the first warehouse, and it costs \$16 to ship a TV to the second warehouse. How many TVs should be shipped to each warehouse to minimize the cost?

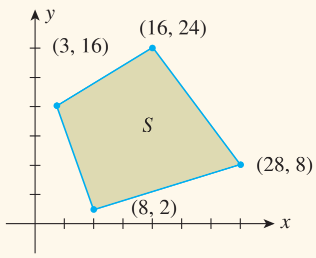
CHAPTER 3 Before Moving On . . .

1. Determine graphically the solution set for the following systems of inequalities.

a. $$\begin{aligned} 2x + y &\le 10 \\ x + 3y &\le 15 \\ x &\le 4 \\ x \ge 0, y &\ge 0 \end{aligned}$$

b. $$\begin{aligned} 2x + y &\ge 8 \\ 2x + 3y &\ge 15 \\ x &\ge 0 \\ y &\ge 2 \end{aligned}$$

2. Find the maximum and minimum values of $Z = 3x - y$ on the following feasible set.



3. Maximize $P = x + 3y$
subject to
$$\begin{aligned} 2x + 3y &\le 11 \\ 3x + 7y &\le 24 \\ x \ge 0, y &\ge 0 \end{aligned}$$

4. Minimize $C = 4x + y$
subject to
$$\begin{aligned} 2x + y &\ge 10 \\ 2x + 3y &\ge 24 \\ x + 3y &\ge 15 \\ x \ge 0, y &\ge 0 \end{aligned}$$

5. Sensitivity Analysis Consider the following linear programming problem:

Maximize $P = 2x + 3y$
subject to
$$\begin{aligned} x + 2y &\le 16 \\ 3x + 2y &\le 24 \\ x \ge 0, y &\ge 0 \end{aligned}$$

a. Solve the problem.
b. Find the range of values that the coefficient of x can assume without changing the optimal solution.
c. Find the range of values that resource 1 (requirement 1) can assume.
d. Find the shadow price for resource 1.
e. Identify the binding and nonbinding constraints.