Complex Analysis Qualifying Exam, August 2023

$\mathbb{D}$ denotes the unit disc $B(0,1)$.

Problem 1: Let $f(z) = \frac{z+1}{z^2(z-i)}$. Give all Laurent series expansions centered at i that converge to f in their domain of convergence.

Problem 2: Using residue calculus, compute

$$\int_0^\infty \frac{dx}{(x^2+1)x^{1/3}} \,.$$

Problem 3: Let $\Omega = \{1 < |z-2| < 3\}$.

a) Show that the function element $(\mathbb{D}, f(z) = \sum_{n=1}^\infty z^n/n\,)$ admits unrestricted analytic continuation in Ω.

b) Show that there does not exist an analytic function g in Ω with $g = f$ on $\mathbb{D}$.

Problem 4: Let f be analytic on $\mathbb{D}$. Suppose there is an annulus $A = \{r < |z| < 1\}$ such that the restriction of f to A is one–to–one. Show that f is one–to–one on $\mathbb{D}$.

Problem 5: Let u be a positive harmonic function on the crescent between the circles $\{|z-i| = 1\}$ and $\{|z-2i| = 2\}$. Assume that $u(z) \to 0$ as $z \to 0$ on the circle $\{|z-(3/2)i| = (3/2)\}$. Show that then $u(z) \to 0$ as $z \to 0$ on any circle $\{|z-ai| = a\}$, $1 < a < 2$.

Problem 6: Construct an entire function that has simple zeros on the positive real axis at the points $\sqrt{n}$, $n = 1, 2, \cdots$, and zeros of order two on the positive imaginary axis at the points $i\sqrt{n}$, $n = 1, 2, \cdots$, and no other zeros.

Problem 7: Given is a point $c \in \mathbb{D}$ and a radius r, $0 < r < (1-|c|)$. Denote by K the compact set $\overline{\mathbb{D}} \setminus \{|z-c| < r\}$. By considering $\int_{|z|=1}(\overline{z} - f(z))dz - \int_{|z-c|=r}(\overline{z} - f(z))dz$, show that $\max_{z\in K} |\overline{z} - f(z)| \geq (1-r)$ for every rational function f with poles off K.

Problem 8: Let f be analytic and bounded in $\mathbb{D}$, let $\{a_k\}_{k=1}^\infty$ be a sequence of (distinct) points in $\mathbb{D}$ such that $\sum_{k=1}^\infty (1-|a_k|) = \infty$. Show that if $f(a_k) = 0$ for all k, then $f \equiv 0$.

Problem 9: Let $\mathcal{F}$ be the family of analytic functions on $\mathbb{D}$ that are one–to–one and omit zero. Show that $\mathcal{F}$ is a normal family in $C(\mathbb{D}, \mathbb{C}_\infty)$.

Problem 10: State the Hadamard Factorization Theorem and explain why it is a "factorization theorem".