1. [12] A subgroup $H$ of a group $G$ is **characteristic** if $\varphi(H) = H$ for any automorphism $\varphi$ of $G$. Show that a characteristic subgroup is normal. Suppose that $G = HK$, where $H$ and $K$ are characteristic subgroups of $G$ with $H \cap K = \{e\}$. Show that $\text{Aut}(G) \simeq \text{Aut}(H) \times \text{Aut}(K)$. (Here, $\text{Aut}(L)$ is the group of automorphisms of $L$.)

2. [12] Show that any group of order $2014 = 2 \cdot 19 \cdot 53$ has a normal cyclic subgroup of index 2. Use this to classify all groups of order 2014.

3. [10] Prove that a finite integral domain is a field. Prove that every prime ideal in a finite commutative ring is maximal.

4. [14] Let $R$ be a commutative ring. Observe that for any two $R$-modules $M, N$, the collection $\text{Hom}(M, N)$ of $R$-module homomorphisms $\varphi : M \to N$ is naturally an $R$-module. Suppose that

$$0 \rightarrow L \xrightarrow{e} M \xrightarrow{f} N \xrightarrow{g} P \rightarrow 0$$

is an exact sequence of $R$-modules (so that $g$ is a surjection whose kernel is equal to the image $f(M)$ of $M$ under $f$, and $e$ is an injection whose image is the kernel of $f$). Let $A$ be an $R$-module. Prove that the induced sequence

$$0 \rightarrow \text{Hom}(A, L) \xrightarrow{e*} \text{Hom}(A, M) \xrightarrow{f*} \text{Hom}(A, N)$$

is exact in that $e*$ is injective and its image is the kernel of the map $f*$. Also prove that the induced sequence

$$\text{Hom}(M, A) \xleftarrow{f^*} \text{Hom}(N, A) \xleftarrow{g^*} \text{Hom}(P, A) \xleftarrow{0}$$

is exact in that $g^*$ is injective and its image is the kernel of the map $f^*$.

5. [10] Let $M$ be an invertible $n \times n$ matrix with real number entries and positive determinant. Show that $M$ can be written as $RK$ where $R$ is in $SO(n)$ ($R$ is orthogonal with determinant 1) and $K$ is an upper triangular matrix with positive entries on the diagonal. Hint: Orthogonal matrices have orthonormal column vectors.
6. [16] Consider a finite field $F$ with $q = p^n$ elements, where $p$ is a prime number and $n$ is a positive integer.
   (a) Explain why every element of $F$ is a root of the polynomial $x^{p^n} - x$.
   (b) Show that if $r$ divides $p^n - 1$ then all the roots of the polynomial $x^r - 1$ lie in $F$.
   (c) Show that the polynomial $x^4 + 1$ is reducible over any finite field. (Hint: It is enough to show it over the prime fields with $p$ elements. Consider the cases $p = 2$ and $p$ odd separately and observe that for $p$ odd, $p^2 - 1$ is congruent to $0$ mod $8$, and $x^8 - 1 = (x^4 - 1)(x^4 + 1)$.)

7. [14] Let $f(x) = x^4 - 4x^2 + 2 \in \mathbb{Q}[x]$, let $E$ be its splitting field contained in $\mathbb{C}$, and let $G$ be the Galois group of $E$ over $\mathbb{Q}$. Without simply citing a theorem about Galois groups of quartic polynomials, prove that $G$ is isomorphic to $\mathbb{Z}/4\mathbb{Z}$. Find a generator for $G$ and determine how it acts on the roots of $f(x)$. It may help to first identify an intermediate subfield $F$, where $\mathbb{Q} \subseteq F \subseteq E$.

8. [12] Let $p$ and $q$ be prime numbers.
   (a) Define a surjective map $\phi : \mathbb{Q}(\sqrt{p}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{q}) \to \mathbb{Q}(\sqrt{p}, \sqrt{q})$ that is both $\mathbb{Q}$-linear and a ring homomorphism.
   (b) If $p$ and $q$ are distinct, show that $\phi$ is an isomorphism.
   (c) If $p = q$, what is a $\mathbb{Q}$-basis for the kernel of $\phi$?