Cover Sheet – Applied Analysis Part

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do not interpret the problem so that it becomes trivial.

Name__________________________
Combined Applied Analysis/Numerical Analysis Qualifier
Applied Analysis Part
August 9, 2012

Instructions: Do any 3 of the 4 problems in this part of the exam. Show all of your work clearly. Please indicate which of the 4 problems you are skipping.

Problem 1. Let $\psi_j$ and $\phi_j$, $j = 1, \ldots, n$, be in $L^2[0, 1]$. Assume the sets $\{\psi_j\}_{j=1}^n$ and $\{\phi_j\}_{j=1}^n$ are linearly independent. Consider the kernel $\kappa(x, y) = \sum_{j=1}^n \psi_j(x) \phi_j(y)$.

(a) Define the term compact operator.
(b) Show that the operator $Ku = \int_0^1 \kappa(\cdot, y)u(y)dy$ is compact on $L^2[0, 1]$.
(c) State and sketch a proof for the Fredholm alternative for compact operators on a Hilbert space.
(d) With $K$ as in part (b), show that the equation $(I - \lambda K)u = f$ has an $L^2$-solution for all $f \in L^2[0, 1]$ if and only if $1/\bar{\lambda}$ is not an eigenvalue of the matrix $A$, where $A_{jk} = \langle \phi_j, \psi_k \rangle$.

Problem 2. Find the first term of the asymptotic series for $F(x) := \int_x^\infty t^2 e^{-t} dt$, $x \to +\infty$.

Problem 3. Let $n > 2$ be an integer and let $x_j = j/n$, $j = 0, \ldots, n$. Consider the functional $J[y] = \frac{1}{2} \int_0^1 (y'')^2 dx$. The admissible functions are in $C^1[0, 1]$. On each closed interval $[x_j, x_{j+1}]$, they are in $C^4[x_j, x_{j+1}]$, for $j = 0, \ldots, n - 1$. Finally, for each $j$, $y(x_j) = y_j$ is fixed.

(a) Assume that the functional is Fréchet differentiable. Show that for $\eta \in C^2[0, 1]$, $\eta(x_j) = 0$, $j = 0, \ldots, n$, one has

$$\Delta J[y, \eta] = \int_0^1 y^{(iv)} \eta dx + \sum_{j=0}^{n-1} y'' \eta' \bigg|_{x_j}^{x_{j+1}}.$$  

(b) If the minimizer $y$ of $J$ exists, use the result above to show that $y$ is a piecewise cubic spline that is in $C^2[0, 1]$.

Problem 4. Let $f \in L^2(\mathbb{R})$. Use the following formulas for the Fourier transform and its inverse:

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt$$  

and $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t} d\omega$.

(a) Define the term band-limited function.
(b) Show that if $f$ is band-limited, then it is infinitely differentiable on $\mathbb{R}$. (Actually, it’s analytic.)
(c) State and prove the the Shannon Sampling Theorem.
Cover Sheet – Numerical Analysis Part

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Name_________________________________________
Problem 1. Consider the variational problem: find $u \in H^1(\Omega)$, such that $a(u, v) = L(v)$ for all $v \in H^1(\Omega)$, where $\Omega = (0, 1) \times (0, 1)$, $\Gamma$ is its boundary, and

\begin{equation}
(1.1) \quad a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx \, dy + \int_0^1 u(s, 0)v(s, 0) \, ds \quad \text{and} \quad L(v) = \int_{\Gamma} g v \, ds.
\end{equation}

Let $V_h \subset H^1(\Omega)$ be a finite dimensional space of conforming piece-wise linear finite elements (Courant triangles) over regular partition of $\Omega$ into triangles. For continuous $v, w$ defined on $\bar{\Gamma} \subseteq \Gamma$, let the bilinear form $Q_e(\Gamma, v, w)$ come from the quadrature

\begin{equation}
(1.2) \quad Q_e(\Gamma, v, w) = \sum_{e \subseteq \Gamma} |e| \left( v(P_e^1)w(P_e^1) + v(P_e^2)w(P_e^2) \right) \approx \int_{\Gamma} vw \, ds.
\end{equation}

Here $e$ is an edge of the triangulation of length $|e|$ with end points $P_e^1$ and $P_e^2$. Consider the FEM: find $u_h \in V_h$ such that

\begin{equation}
(1.3) \quad a_h(u_h, v) = L_h(v), \quad \forall v \in V_h,
\end{equation}

where $a_h(u_h, v)$ and $L_h(v)$ are defined from $a(u, v)$ and $L(v)$ with the boundary integrals approximated using quadrature (1.2).

Complete the following tasks:

(a) **Derive** the strong form to the problem (1.1).
(b) **Prove** that the bilinear form $a(u, v)$ is coercive on $H^1$.
(c) **Prove** that for $\bar{\Gamma} = \{(x, 0), 0 < x < 1\}$, there are constants $c_1$ and $c_2$, independent of $h$, such that

$$c_1 Q_e(\Gamma, v, v) \leq \int_0^1 v(x, 0)^2 \, dx \leq c_2 Q_e(\Gamma, v, v), \quad \forall v \in V_h.$$ 

Note that this inequality and part (b) immediately imply

$$a_h(v, v) \geq \alpha \|v\|^2_{H^1(\Omega)}, \quad \forall v \in V_h$$

for some $\alpha > 0$ independent of $h$.
(d) Apply Strang’s First Lemma to estimate the error in $H^1$-norm for the FEM (1.3). You may assume that $g$ is as regular (smooth) as needed by your analysis and you can use (without proof) standard approximation properties for the finite element space $V_h$.

Problem 2. Consider the following initial boundary value problem: find $u(x, t)$ such that

\begin{equation}
(2.1) \quad \frac{\partial}{\partial t}(u - \Delta u) - \mu \Delta u = f, \quad x \in \Omega, \quad T \geq t > 0,
\end{equation}

$$u(x, t) = 0, \quad x \in \partial \Omega, \quad T \geq t > 0,$$

$$u(x, 0) = u_0(x), \quad x \in \Omega,$$
where $\Omega$ is a polygonal domain in $\mathbb{R}^2$, $\mu > 0$ is a given constant, and $f(x,t)$ and $u_0(x)$ are given right hand side and initial data functions.

(a) **Derive** a weak formulation of this problem and derive an *a priori* estimate for the solution in the norm

$$
\|u(t)\|_{H^1(\Omega)} = \left(\|u(t)\|_{L^2(\Omega)}^2 + \|\nabla u(t)\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}
$$

in terms of the right-hand side and the initial data.

(b) **Write down** the fully discrete scheme based on implicit (backward) Euler approximation in time and the finite element method in space with continuous piece-wise linear functions. **Prove** unconditional stability in the $H^1$-norm for the resulting approximation.

(c) Consider now the forward Euler approximation for the derivative in $t$. **Find** the Courant condition for stability of the resulting method in a norm of your choice.

**Problem 3.** Let $T_h$ be a partition of $(0,1)$ into finite elements of equal size $h = 1/N$, $N > 1$ an integer, and $x_i = ih$, $i = 0, 1, \ldots, N$. Consider the finite dimensional space $V_h$ of continuous **piece-wise quadratic** functions on $T_h$. The degrees of freedom on finite element $(x_{i-1}, x_i)$ are

$$
\{ v(x_{i-1}), v(x_i), \frac{1}{h} \int_{x_{i-1}}^{x_i} v \, dx \}.
$$

(1) **Explicitly find the nodal basis** of $V_h$ over the finite element $(x_{i-1}, x_i)$, corresponding to these degrees of freedom.

(2) **Prove** that

$$
\sup_{\phi \in H^1(0,1)} \frac{\int_0^1 (u - \Pi_h u) \phi \, dx}{\|\phi\|_{H^1(0,1)}} \leq C h \|u - \Pi_h u\|_{L^2(0,1)}, \quad \forall u \in H^1(0,1).
$$

Here $\Pi_h u$ is the finite element interpolant of $u$ with respect to the nodal basis of $V_h$ defined by (3.1).