Cover Sheet – Applied Analysis Part

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do not interpret the problem so that it becomes trivial.

Name______________________________________________________________
Instructions: Do any 3 of the 4 problems in this part of the exam. Show all of your work clearly. Please indicate which of the 4 problems you are skipping.

Notation: $\mathcal{H}$ denotes a complex, separable Hilbert space, with inner product and norm given by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. $\mathcal{B}(\mathcal{H})$ and $\mathcal{C}(\mathcal{H})$ are, respectively, the set of bounded linear operators on $\mathcal{H}$ and the set of compact linear operators on $\mathcal{H}$.

Problem 1. This problem is aimed at proving the Riemann-Lebesgue Lemma: If $f \in L^1[0,1]$, then $\lim_{\lambda \to \infty} \int_0^1 f(x)e^{i\lambda x}dx = 0$.

(a) Show that if $p(x) = \sum_{k=0}^n a_k x^k$, then $\lim_{h \to 0} \phi_h = \phi''$.

(b) Let $g \in C[0,1]$, $\lim_{\lambda \to \infty} \int_0^1 g(x)e^{i\lambda x}dx = 0$.

(c) Use (a), (b) and the density of $C[0,1]$ in $L^1$ to complete the proof.

Problem 2. Let $D$ be the set of compactly supported $C^\infty$ functions defined on $\mathbb{R}$ and let $D'$ be the corresponding set of distributions.

(a) Define convergence in $D$ and $D'$.

(b) Let $\phi \in D$ and define $\phi_h(x) := (\phi(x+h) - 2\phi(x) + \phi(x-h))/h^2$. Show that, in the sense of $D$, $\lim_{h \to 0} \phi_h = \phi''$.

(c) Let $T \in D'$ and define $T_h = (T(x+h) - 2T(x) + T(x-h))/h^2$. Show that, in the sense of distributions, $\lim_{h \to 0} T_h = T''$.

Problem 3. Let both $K \in \mathcal{C}(\mathcal{H})$ and $L \in \mathcal{B}(\mathcal{H})$ be self-adjoint.

(a) Show that $\|L\| = \sup_{\|u\|=1} |\langle Lu, u \rangle|$. (Hint: look at $\langle L(u + v), u + v \rangle - \langle L(u - v), u - v \rangle$.)

(b) Prove this: Either $\|K\|$ or $-\|K\|$ is an eigenvalue of $K$.

Problem 4. Let $L$ be a (possibly unbounded) closed, densely defined linear operator with domain $D_L \subseteq \mathcal{H}$.

(a) Define these: the resolvent set, $\rho(L)$; the discrete spectrum, $\sigma_d(L)$; the continuous spectrum, $\sigma_c(L)$; and the residual spectrum, $\sigma_r(L)$.

(b) Show that $L^*$, the adjoint of $L$, is closed and densely defined.

(c) Show that if $L$ is self-adjoint, then $\sigma_r(L) = \emptyset$. 

Cover Sheet – Numerical Analysis Part

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Name:__________________________________________________________
NUMERICAL ANALYSIS QUALIFIER
August 11, 2015

In the problems below \( \mathbb{P}^j \) denotes the space of polynomials on \( \mathbb{R}^2 \) of degree at most \( j \).

**Problem 1.** Let \( T \subset \mathbb{R}^2 \) be a triangle with vertices \( v_1, v_2, \) and \( v_3 \). Let \( p_1 = (v_1 + v_2 + v_3)/3, \)
\( p_2 = (2v_1 + v_2)/3, \) \( p_3 = (2v_1 + v_3)/3, \) \( p_4 = v_2, \)
\( v_5 = (v_2 + v_3)/2, \) and \( p_6 = v_3. \) Given \( q \in \mathbb{P}^2, \)
let \( \sigma_i(q) = q(p_i) \).

1. Show that the triple \((T, \mathbb{P}^2, \Sigma)\) constitutes a finite element, where \( \Sigma = \{ \sigma_i \}_{i=1}^6 \).
2. Write down the nodal basis function \( \phi_1 \) corresponding to this finite element. That is, \( \phi_1 \in \mathbb{P}^2 \) should satisfy \( \phi_1(p_1) = 1 \) and \( \phi_1(p_j) = 0, j \neq 1. \)

*Hint:* You should use barycentric (area) coordinates to derive your solution.

**Problem 2.** For \( f \in L^2(0,1), \) consider the following weak formulation: Seek \((u, v) \in V := H_0^1(0,1) \times H_0^1(0,1) \)
satisfying for all \((\phi, \psi) \in V \)
\[
(2.1) \quad a((u, v); (\phi, \psi)) := \int_0^1 u'\phi' + \int_0^1 v'\psi' - \int_0^1 v\phi = \int_0^1 f\psi =: L(\psi).
\]

1. What is the corresponding strong form satisfied by \( u \) (eliminate \( v \))?
2. Show that for all \( w \in H_0^1(0,1) \)
\[
\left( \int_0^1 w^2 \right)^{1/2} \leq \left( \int_0^1 |w'|^2 \right)^{1/2}.
\]
3. Using Part (2) show that \( a(\cdot, \cdot) \) coerces the natural norm on \( V \):
\[
|||\phi, \psi||| := \left( ||\phi||_{H^1(0,1)}^2 + ||\psi||_{H^1(0,1)}^2 \right)^{1/2}
\]
and explicitly find the coercivity constant.
4. Let \( V_h \) be a finite dimensional subspace of \( V \). Explain why there is a unique \((u_h, v_h) \in V_h \)
satisfying for all \((\phi_h, \psi_h) \in V_h \)
\[
a((u_h, v_h); (\phi_h, \psi_h)) = L(\psi_h).
\]
5. Show that
\[
|||u - u_h, v - v_h||| \leq C_1 \inf_{(\phi_h, \psi_h) \in V_h} |||u - \phi_h, v - \phi_h|||
\]
(find \( C_1 \) explicitly).
6. You may assume that \( u, v \in H_0^1(0,1) \cap H^2(0,1) \). Propose a discrete space \( V_h \) such that
\[
|||u - u_h, v - v_h||| \leq C_2 h (||u||_{H^2(0,1)} + ||v||_{H^2(0,1)})
\]
for a constant \( C_2 \) independent of \( h \). Justify your suggestion.

**Problem 3.** For \( \Omega = (0,1)^2 \) and \( u_0 \in L^2(\Omega) \), consider the parabolic problem:
\[
u_t - \Delta u + (u_x + u_y) = 0, \quad (x, t) \in \Omega \times (0, T],
\]
(3.1)
\[
u(x, t) = 0, \quad x \in \partial \Omega, t \in (0, T],
\]
\[
u(x, 0) = u_0(x), \quad x \in \Omega.
\]
1. Using a finite element space $V_h \subset H^1_0(\Omega)$, derive a semi-discrete approximation to (3.1) having solution $u_h(t) \in V_h$. This approximation satisfies $u_h(0) = \pi_h u_0$ with $\pi_h$ denoting the $L^2(\Omega)$-projection onto $V_h$.

2. Show that 
\[ \|u_h(t)\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)}, \quad t \in [0,T]. \]

*Hint:* Recall the integration-by-parts formula 
\[ \int_{\Omega} uv x_i \, dx = \int_{\partial \Omega} uv \nu_i \, d\sigma - \int_{\Omega} u x_i v \, dx, \]

for $u, v \in H^1(\Omega)$, where $\nu_i$ is the $i$-th component of the outward unit normal on $\partial \Omega$.

3. Consider the initial value problem:
\[ w' + \lambda w = 0, \quad w(0) = w_0, \]

and the time stepping method with step size $k$:
\[ \frac{w^{n+1} - w^n}{k} + \lambda (\theta w^{n+1} + (1 - \theta) w^n) = 0. \]

Here $\theta$ is a parameter in $[0,1]$ and $\lambda \in \mathbb{R}$ with $\lambda > 0$. Use this method to develop a fully discrete ($\theta$ dependent) approximation to (3.1) (Note: $\theta = 1$ and $\theta = 0$ correspond to, respectively, backward and forward Euler time stepping).

4. Let $U^n \in V_h$ be the resulting fully discrete approximation after $n$ steps using $U^0 = \pi_h u_0$. Show that for $\theta \in [1/2, 1]$,
\[ \|U^n\|_{L^2(\Omega)} \leq \|U^0\|_{L^2(\Omega)}. \]

*Hint:* Test with a discrete function that depends on $\theta$. 
