1. Give the statements of the following:
   (a) Runge’s theorem;
   (b) the Schwarz lemma.

2. Find the Laurent expansion for the function $z^{-1}(z + 1)^{-1}$ centered at $a = 1$ in $\{1 < |z - 1| < 2\}$.

3. Let $f$ be a meromorphic function in $\mathbb{C}$. Suppose that $f$ is doubly periodic, i.e. that for some non-zero numbers $a, b \in \mathbb{C}$,
   
   \[ f(z) = f(z + a) \quad \text{and} \quad f(z) = f(z + b) \]

   for any $z \in \mathbb{C}$. Consider the parallelogram $P$ with sides $(0, a)$ and $(0, b)$. Assuming that $f$ does not have any zeros or poles on the sides of $P$, prove that the number of zeros of $f$ in $P$ is equal to the number of poles of $f$ in $P$.

4. Apply the Residue Theorem to evaluate the integral
   
   \[ \int_{-\infty}^{\infty} \frac{\cos x \, dx}{(x^2 + 1)(x^2 + 4)}. \]

5. Let $u(z)$ be a real continuous function on the unit circle. Write an integral formula for the analytic function $f(z)$ in the unit disk such that
   
   \[ \lim_{z \to \xi} \Re f(z) = u(\xi) \]

   for every $\xi$ on the unit circle.

6. Let $f$ be an analytic function in the unit disk $\mathbb{D}$. Suppose that $|f(z)| \leq 1$ in $\mathbb{D}$. Prove that then

   \[ \frac{|f(0)| - |z|}{1 - |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 + |f(0)||z|}. \]

7. Show that the equation

   \[ e^z - z = \lambda, \]

   where $\lambda > 1$, has exactly one root in the left half-plane $\{\Re z < 0\}$.

8. Let $f$ be a function, analytic and bounded in the strip $\{|\Im z| < \pi/2\}$. Suppose that $f(\ln n) = 0$ for all $n \in \mathbb{N}$. Prove that $f$ is identically zero.

9. Let $f(z)$ be a meromorphic function in $\mathbb{C}$. Show that $f$ can be expressed as $f(z) = g(z)/h(z)$ where $g$ and $h$ are entire.

10. Describe the set of all harmonic functions $u(x, y)$ in $\mathbb{C}$ such that the product $(x^2 - y^2)u(x, y)$ is harmonic in $\mathbb{C}$.