Complex Analysis
Qualifying Examination
August 2015

1. Find every complex number $z$ for which the infinite series
$$
\sum_{n=1}^{\infty} \left( \frac{2015 + i}{2015 - i} \right)^n \left( \frac{z - 2015}{z + 2015} \right)^n
$$
converges.

2. Determine every complex number $w$ that can be written in the form $\sin(z)$ for some complex
number $z$ having positive imaginary part. In other words, what is the image of the open
upper half-plane under the sine function?

3. Prove that
$$
\int_0^{\infty} \frac{(\log x)^2}{1 + x^2} \, dx = \frac{\pi^3}{8}.
$$

4. When $n$ is an integer, the Bessel function $J_n(z)$ can be defined to be the coefficient of $t^n$ in
the Laurent series about the origin of
$$
\exp \left( \frac{1}{2} z \left( t - \frac{1}{t} \right) \right)
$$
(series with respect to the variable $t$). Use this definition to show that $J_{-n}(z) = (-1)^n J_n(z)$.

5. When the variable $z$ is restricted to the first quadrant (where $\text{Re} \, z > 0$ and $\text{Im} \, z > 0$), how
many zeroes does the polynomial $z^{2015} + 8z^{12} + 1$ have?

6. Suppose $f$ is an entire function such that $f(x + 0i)$ is real for every real number $x$, and
$f(0 + yi)$ is real for every real number $y$. Prove the existence of an entire function $g$ such
that $f(z) = g(z^2)$ for every complex number $z$.

7. Does there exist a holomorphic function that maps the open unit disk surjectively (but not
injectively) onto the whole complex plane?

8. Determine the group of holomorphic bijections (automorphisms) of \{ $z \in \mathbb{C} : |z| > 1$ \},
the complement of the closed unit disk.

9. On the punctured plane $\mathbb{C} \setminus \{0\}$, can the function $e^{1/z}$ be obtained as the pointwise limit of
a sequence of polynomials in $z$?

10. Prove that if $f_1$ and $f_2$ are holomorphic functions with no common zero in a region of the
complex plane, then there exist holomorphic functions $g_1$ and $g_2$ such that $f_1g_1 + f_2g_2$ is
identically equal to 1 in the region.
[Exactly 100 years ago, the algebraist J. H. M. Wedderburn proved this proposition by
applying Mittag-Leffler’s theorem.]