Work as many of these ten problems as you can in four hours. Start each problem on a new sheet of paper.

#1. Let \( f \) be a Lebesgue integrable, real–valued function on \((0,1)\) and for \( x \in (0,1) \) define

\[
g(x) = \int_x^1 t^{-1} f(t) \, dt.
\]

Show that \( g \) is Lebesgue integrable on \((0,1)\) and that \( \int_0^1 g(x) \, dx = \int_0^1 f(x) \, dx \).

#2. Let \( f_n \in C[0,1] \). Show that \( f_n \to 0 \) weakly if and only if the sequence \( (\|f_n\|)_{n=1}^\infty \) is bounded and \( f_n \) converges pointwise to 0.

#3. Let \( (X, \mu) \) be a measure space with \( 0 < \mu(X) \leq 1 \) and let \( f : X \to \mathbb{R} \) be measurable. State the definition of \( \|f\|_p \) for \( p \in [1,\infty] \). Show that \( \|f\|_p \) is a monotone increasing function of \( p \in [1,\infty) \) and that \( \lim_{p \to \infty} \|f\|_p = \|f\|_\infty \).

#4. (a) Is there a signed Borel measure \( \mu \) on \([0,1]\) such that

\[
p'(0) = \int_0^1 p(x) \, d\mu(x)
\]

for all real polynomials \( p \) of degree at most 19?

(b) Is there a signed Borel measure \( \mu \) on \([0,1]\) such that

\[
p'(0) = \int_0^1 p(x) \, d\mu(x)
\]

for all real polynomials \( p \)?

(Justify your answers).

#5. Let \( \mathcal{F} \) be the set of all real–valued functions on \([0,1]\) of the form

\[
f(t) = \frac{1}{\prod_{j=1}^n (t - c_j)}
\]

for natural numbers \( n \) and for real numbers \( c_j \notin [0,1] \). Prove or disprove: for all continuous, real–valued functions \( g \) and \( h \) on \([0,1]\) such that \( g(t) < h(t) \) for all \( t \in [0,1] \), there is a function \( a \in \text{span} \mathcal{F} \) such that \( g(t) < a(t) < h(t) \) for all \( t \in [0,1] \).

#6. Let \( k : [0,1] \times [0,1] \to \mathbb{R} \) be continuous and let \( 1 < p < \infty \). For \( f \in L^p[0,1] \), let \( Tf \) be the function on \([0,1]\) defined by

\[
(Tf)(x) = \int_0^1 k(x,y) f(y) \, dy.
\]

Show that \( Tf \) is a continuous function on \([0,1]\) and that the image under \( T \) of the unit ball in \( L^p[0,1] \) has compact closure in \( C[0,1] \).
#7. (a) Define the total variation of a function \( f : [0, 1] \to \mathbb{R} \) and absolute continuity of \( f \).
(b) Suppose \( f : [0, 1] \to \mathbb{R} \) is absolutely continuous and define \( g \in C[0, 1] \) by
\[
g(x) = \int_0^1 f(xy) \, dy.
\]
Show that \( g \) is absolutely continuous.

#8. (a) State the definition of absolute continuity, \( \nu \ll \mu \), for positive measures \( \mu \) and \( \nu \), and state the Radon–Nikodym Theorem, (or the Lebesgue–Rado–Nikodym Theorem, if you prefer.)
(b) Suppose that we have \( \nu_1 \ll \mu_1 \) and \( \nu_2 \ll \mu_2 \) for positive measures \( \nu_i \) and \( \mu_i \) on measurable spaces \((X_i, \mathcal{M}_i), (i = 1, 2)\). Show that we have \( \nu_1 \times \nu_2 \ll \mu_1 \times \mu_2 \), and
\[
d(\nu_1 \times \nu_2)(x, y) = \frac{d\nu_1(x)}{d\mu_1(x)} \frac{d\nu_2(y)}{d\mu_2(y)}.
\]

#9. (a) Let \( E \) be a nonzero Banach space and show that for every \( x \in E \) there is \( \phi \in E^* \) such that \( \|\phi\| = 1 \) and \( |\phi(x)| = \|x\| \).
(b) Let \( E \) and \( F \) be Banach spaces, let \( \pi : E \to F \) be a bounded linear map and let \( \pi^* : F^* \to E^* \) be the induced map on dual spaces. Show that \( \|\pi^*\| = \|\pi\| \).

#10. Let \( X \) be a real Banach space and suppose \( C \) is a closed subset of \( X \) such that
\[
(i) \ x_1 + x_2 \in C \text{ for all } x_1, x_2 \in C,
(ii) \ \lambda x \in C \text{ for all } x \in C \text{ and } \lambda > 0,
(iii) \text{ for all } x \in X \text{ there exist } x_1, x_2 \in C \text{ such that } x = x_1 - x_2.
\]
Prove that, for some \( M > 0 \), the unit ball of \( X \) is contained in the closure of
\[
\{x_1 - x_2 \mid x_i \in C, \|x_i\| \leq M, (i = 1, 2)\}.
\]
Deduce that every \( x \in X \) can be written \( x = x_1 - x_2 \), with \( x_i \in C \) and \( \|x_i\| \leq 2M\|x\| \), \((i = 1, 2)\).