Notation. Let \( \mathbb{R}^n \) denote real \( n \)-space. Employ the summation convention: any repeated index appearing as a subscript and superscript is summed over.

**Show your work.**

1.) Let \( C \) be a subset of a topological space \( X \).
   (a) Prove that if \( C \) is connected, then the closure of \( C \) is connected.
   (b) Prove or give a counter-example to the following statement: if \( C \) is connected, then the interior of \( C \) is connected.

2.) Prove that a countable product of separable spaces is separable.

3.) (a) Is the set of rational numbers \( \mathbb{Q} \) (as a subspace of \( \mathbb{R} \)) locally compact? Prove your answer.
   (b) Prove that if a topological space \( X \) is locally compact, Hausdorff, and second countable, then it is metrizable.

4.) Let \( f : X \to Y \) be a continuous map between topological spaces \( X \) and \( Y \).
   (a) Define what it means for \( f \) to be a quotient (an identification) map.
   (b) Prove that if the map \( f : X \to Y \) is open and onto, then \( f \) is a quotient map.
   (c) Let \( C \) be the union of the \( x \)-axis and the \( y \)-axis of \( \mathbb{R}^2 \) and define \( g : \mathbb{R}^2 \to C \) as follows:
      \[
      g(x, y) = \begin{cases} 
      (x, 0) & \text{if } x \neq 0 \\
      (0, y) & \text{if } x = 0
      \end{cases}
      \]
      Does the quotient topology on \( C \) induced by \( g \) coincide with the subspace topology on \( C \) induced from the standard topology of \( \mathbb{R}^2 \)? Prove your answer.

5.) (a) Give the definition of a paracompact space.
   (b) Using the definition of paracompactness only, prove that \( \mathbb{R}^n \) (with the standard topology) is paracompact.
   (c) Give an example to show that if \( X \) is paracompact, it does not follow that for every open covering of \( \mathcal{A} \) of \( X \) there is locally finite subcollection of \( \mathcal{A} \) that covers \( X \).

6.) Let \( \{K_\alpha\}_{\alpha \in A} \) be a collection of compact subsets of a Hausdorff space \( X \) which is closed with respect to finite intersections. Let \( K = \bigcap_{\alpha \in A} K_\alpha. \)
   (a) Suppose that \( W \) is an open subset of \( X \) such that \( K \subset W \). Prove that \( K_\alpha \subset W \) for some \( \alpha \in A \).
(b) Prove that if $K_\alpha$ is connected for each $\alpha \in A$, then $K$ is connected.

7.) (a) State the definition of a smooth $n$-dimensional manifold.
(b) Define $F : \mathbb{R}^3 \to \mathbb{R}^1$ by $F(x, y, z) = x \cos(z) + y \sin(z)$. Prove that the level set $F^{-1}(0)$ is a smooth 2-dimensional manifold.

8.) Let $X_1, \ldots, X_m$ be linearly independent vector fields on $\mathbb{R}^n$, $m \leq n$. Fix the index ranges

\begin{align*}
1 & \leq a, b \leq m \\
1 & \leq i, j \leq n \\
m + 1 & \leq s, t \leq n.
\end{align*}

Prove the following.
(a) For every $p \in \mathbb{R}^n$ there exists an open set $U \subset \mathbb{R}^n$ containing $p$ and linearly independent 1-forms $\eta^1, \ldots, \eta^n$ on $U$ with the property that $\eta^i(X_a) = \delta^i_a$. Here $\delta^i_a$ is the Kronecker delta.
(b) Prove that $[X_a, X_b] \subset \text{span}_{\mathbb{R}} \{X_1, \ldots, X_m\}$ if and only if there exist 1-forms $\alpha^s_t$ on $U$ such that $d\eta^s = \alpha^s_t \wedge \eta^t$, for all $m + 1 \leq s \leq n$.

9.) Let $Z = \mathbb{R}^{n+1} \setminus \{0\}$. Define an equivalence relation $\sim$ on $U$ by

$x \sim y$ if and only if there exists $\lambda \neq 0$ such that $y = \lambda x$.

Recall that projective $n$-space is the manifold $\mathbb{P}^n = Z/\sim$. Given $x = (x^0, \ldots, x^n) \in Z$, let $[x] = [x^0 : \ldots : x^n] \in \mathbb{P}^n$ denote the corresponding equivalence class. Fix $p = [1 : 0] \in \mathbb{P}^1$ and $q = [1 : 0 : 0 : 0] \in \mathbb{P}^3$.
(a) Describe a coordinate chart $(U, \varphi)$ about $p$, and coordinate chart $(V, \psi)$ about $q$.
(b) Let $\nu : \mathbb{P}^1 \to \mathbb{P}^3$ be the Veronese map $\nu([s : t]) = [s^3 : s^2 t : st^2 : t^3]$. Give the local coordinate expression of $\nu$ with respect to the coordinates $(U, \varphi)$ and $(V, \psi)$.
(c) Express the push-forward $\nu_* : T_p \mathbb{P}^1 \to T_q \mathbb{P}^3$ in terms of the local coordinates.
(d) Express the pull-back $\nu^* : T_q \mathbb{P}^3 \to T_p \mathbb{P}^1$ in terms of the local coordinates.

10.) Consider a unit speed curve $C : t \mapsto (\alpha(t), 0, \beta(t))$ in $\mathbb{R}^3$ with $\alpha(t) > 0$. Let $S$ be the surface of revolution obtained by rotating $C$ about the $z$-axis. The $(\alpha(t), \beta(t))$ for which $S$ is of constant Gauss curvature $K = -1$ are characterized by an ordinary differential equation. Identify that equation.