Problem 1. Let $X$ be a compact Hausdorff space.

(a) Let $n \geq 1$ and
\[
\{ f_i : X \rightarrow \mathbb{R} \mid i = 1, \ldots, n \}
\]
be a finite family of continuous functions such that, for each pair of distinct points $x, y \in X$, there exists $i, 1 \leq i \leq n$, with $f_i(x) \neq f_i(y)$. Show that $X$ is homeomorphic to a subspace of $\mathbb{R}^n$.

(b) Let $f : X \rightarrow X$ be an injective continuous function. Show that there exists a nonempty closed subset $A$ of $X$ such that $f(A) = A$.

Problem 2. Let $X$ be a topological space. Show that the intersection of any two dense open subsets of $X$ is also dense.

Problem 3. Let $X$ be a locally compact space and let $A$ be a subset of $X$ such that, for every compact subset $K$ of $X$, the intersection $A \cap K$ is a closed subset of $X$. Show that $A$ is a closed subset of $X$.

Problem 4. Consider the equivalence relation $\sim$ on $I = [0,1]$ given by
\[
x \sim y \iff x = y \text{ or } 1/3 < x, y < 2/3,
\]
and the quotient space $X = I/\sim$. Prove or disprove each of the following

(a) $X$ is Hausdorff.
(b) $X$ is connected.
(c) $X$ is compact.
Problem 5. In $\mathbb{R}^3$, set

\[ X_1 = x_1^2 x_2 \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3}, \quad X_2 = 2x_1 \frac{\partial}{\partial x_2}, \quad \omega = x_3 dx_1 \wedge dx_2 + x_2^2 dx_1 \wedge dx_3. \]

(a) Compute $[X_1, X_2]$.
(b) Compute $\omega(X_1, X_2)$.
(c) Compute $\omega \wedge (x_2 dx_2)$.
(d) Compute $d\omega$.
(e) Prove that for any point $p \in \mathbb{R}^3$ there are no neighborhood $U$ and coordinate functions $y_1, y_2, y_3$ on $U$ such that $X_1 = \frac{\partial}{\partial y_1}$ and $X_2 = \frac{\partial}{\partial y_2}$.
(f) On the set $M := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \neq 0\}$ define the distribution $D = \text{span}(X_1, X_2)$. Prove that for any point $p \in M$ there exist a neighborhood $U$, coordinate functions $(y^1, y^2, y^3)$ on $U$, and vector fields $Y_1$ and $Y_2$ on $U$ such that $D = \text{span}(Y_1, Y_2)$ and $Y_i = \frac{\partial}{\partial y_i}$, $i = 1, 2$. Give an example of such vector fields $Y_1, Y_2$ (in the original coordinates $(x_1, x_2, x_3)$).

Problem 6. Define $f : \mathbb{R}^3 \to \mathbb{R}^2$ by $f(x, y, z) = (x^2 + y^2, yz)$. Let $(u, v)$ denote standard coordinates in $\mathbb{R}^2$.

(a) Calculate $f^*(udv + vdu)$.
(b) Calculate $f_* \left( \frac{\partial}{\partial y_1}(10, -5, -1) \right)$.
(c) Find all regular values of $f$.
(d) Find all $(a, b)$ in $\mathbb{R}^2$ such that the set $f^{-1}(a, b)$ is a nonempty embedded submanifold of $\mathbb{R}^3$.

Problem 7. Suppose $M$ is a smooth $n$-dimensional manifold and $D$ is a smooth rank $k$ distribution on $M$. Recall that a $p$-form $\eta$ annihilates $D$ if $\eta(X_1, \ldots, X_p) = 0$ whenever $X_1, \ldots, X_p$ are local sections of $D$. Let $\omega^1, \ldots, \omega^{n-k}$ be smooth local defining forms for $D$ over an open subset $U \subseteq M$, i.e. $D_q = \text{Ker} \omega^1|_q \cap \ldots \cap \text{Ker} \omega^{n-k}|_q$ for $q \in U$. Prove that a smooth $p$-form $\eta$ defined on $U$ annihilates $D$ if and only if it can be expressed in the form

\[ \eta = \sum_{i=1}^{n-k} \omega^i \wedge \beta^i \]

for some smooth $(p-1)$-forms $\beta^1, \ldots, \beta^{n-k}$ on $U$.

Problem 8. Assume that for any $p = (x, y) \in \mathbb{R}^2$ the inner product $\langle \cdot, \cdot \rangle_p$ is given as follows: if $v_1, v_2 \in T_p \mathbb{R}^2$, then $\langle v_1, v_2 \rangle = \lambda(p)(v_1 \cdot v_2)$, where $v_1 \cdot v_2$ is the standard inner product in $\mathbb{R}^2$ and $\lambda : \mathbb{R}^2 \to \mathbb{R}$ is a smooth positive function. Prove that the Gaussian curvature $K$ of the corresponding Riemannian metric is given by

\[ K = -\frac{1}{2\lambda} \Delta (\log(\lambda)), \]

where $\Delta$ is the Laplacian, $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$. 