INSTRUCTIONS:
- There are 8 problems. Work on all of them.
- Prove your assertions.
- Use a separate sheet of paper for each problem and write only on one side of the paper.
- Write your name on the top right corner of each page.

Problem 1. Let $X$ be the interval $[0,1]$ with the following topology. A subset $U$ of $X$ is open if and only if it contains the interval $(0,1)$ or it does not contain the point $1/2$.

(a) Is the topology on $X$ smaller (coarser) than, larger (finer) than, or not comparable to the the standard topology on the unit interval? Please justify your answer.

(b) Determine the closure of the set $\{1/4\}$ in $X$. Please justify your answer.

(c) Show that $X$ is a $T_0$ space, but it is not a $T_1$ space.

Problem 2. Let $X$ be a compact space, $\{C_j \mid j \in J\}$ a nonempty family of closed sets in $X$, $C = \bigcap_{j \in J} C_j$, and $U$ an open set in $X$ containing $C$. Show that there exists a finite subset $\{j_1, j_2, \ldots, j_n\}$ of $J$ such that

$$C_{j_1} \cap C_{j_2} \cap \cdots \cap C_{j_n} \subseteq U.$$

Problem 3. Let $X$ and $Y$ be topological spaces, and $f : X \to Y$ and $g : Y \to X$ be two maps such that, for all $y \in Y$, $f(g(y)) = y$. Show that if $Y$ is connected and $f^{-1}(y)$ is connected for all $y \in Y$, then $X$ is connected.

Problem 4. Let $(X,d)$ be a compact metric space and $f : X \to X$ be a distance preserving map (a map such that, for all $x,y \in X$, $d(f(x), f(y)) = d(x,y)$).

(a) Show that $f$ is injective.

(b) Show that, for every point $x \in X$ and every $\varepsilon$-ball $B_{\varepsilon}(x)$ centered at $x$, one of the balls in the sequence

$$f(B_\varepsilon(x)), \quad f(f(B_\varepsilon(x))), \quad f(f(f(B_\varepsilon(x)))), \quad \ldots$$

has nonempty intersection with $B_\varepsilon(x)$.

(c) Use part (b), or any other method, to prove that $f$ is surjective.
Problem 5. Let $V$ be a real vector space of dimension $n+1$. Define an equivalence relation on $V \setminus \{0\}$ by $u \sim v$ if $u = \lambda v$ for some nonzero $\lambda \in \mathbb{R}$. Let $\mathbb{P}(V) = (V \setminus \{0\})/\sim$ denote the quotient space, equipped with the quotient topology. Prove that $\mathbb{P}(V)$ is a smooth manifold of dimension $n$.

Problem 6. Let $M = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ be the upper half-plane. Let $u \cdot v$ denote the dot product of vectors $u, v \in \mathbb{R}^2$. Use the natural identification $T_{(x,y)}M \simeq \mathbb{R}^2$ to define a metric $g$ on $M$ by

$$g_{(x,y)}(u,v) := \frac{u \cdot v}{y^2} \quad \text{for all } u,v \in T_{(x,y)}M.$$ 

Compute the Gauss curvature of $M$.

Problem 7. Prove that the distribution $\mathcal{D}$ on $\mathbb{R}^3$ spanned by the vector fields

$$X = (1 + z^2) \frac{\partial}{\partial z},$$

$$Y = \frac{\partial}{\partial x} - \frac{\partial}{\partial y} + 4(y - x) \frac{\partial}{\partial z}$$

is involutive. Find flat coordinates for the distribution; that is, find coordinates $(u,v,w)$ on $\mathbb{R}^3$ so that $\mathcal{D}$ is spanned by $\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\}$.

Problem 8. For what values of $c \in \mathbb{R}$ is $\{xyz = c\} \subset \mathbb{R}^3$ a smooth, embedded submanifold? What are the dimensions of these manifolds?