Topology Qualifying Examination
January 2013

Instructions. Answer all questions. Write your name and page number in the upper right corner of each page. Start each problem on a new sheet of paper, and use only one side of each sheet.

Notation. \( \mathbb{N} \) denotes the positive integers. \( \mathbb{R} \) denotes the real numbers. \( \mathbb{R}^n \) denotes Euclidean \( n \)-dimensional space.

1. Let \( X \) be a metric space. Given a cover \( \{U_\alpha\} \) of \( X \) by subsets of \( X \), a \textit{Lebesgue number} for the cover is a number \( \epsilon > 0 \) such that if \( A \subset X \) and \( \text{diam}(A) < \epsilon \), then \( A \) is contained in at least one set \( U_\beta \) of the cover.
   
   (a) Prove that every open cover of a compact metric space \( X \) has a Lebesgue number.
   
   (b) Prove that if \( f : X \to Y \) is a continuous map from a compact space \( X \) to a metric space \( Y \), then \( f \) is uniformly continuous.

2. Let \( X \) and \( Y \) be topological spaces. Let \( f : X \to Y \) be a quotient map. Define \textit{quotient map}.
   
   Show that if \( Y \) is connected and \( f^{-1}(y) \) is connected for all \( y \in Y \), then \( X \) is connected.

3. Define \textit{paracompact space}. Prove that if \( X \) is paracompact, then \( X \) is normal.

4. Let \( X \) and \( Y \) be topological spaces. Let \( f : X \to Y \) be a surjective function satisfying the condition that \( \text{int}(f(A)) \subset f(\text{int}(A)) \) for any subset \( A \subset X \). Show that \( f \) is continuous.

5. For every \( S \subset \mathbb{N} \), let \( X_S = \{0, 1\} \) with the discrete topology, and let \( X = \prod S X_S \) with the product topology. Let \( f_n(S) \) be 0 if \( n \in S \), and 1 if \( n \not\in S \). Prove that the sequence \( \{f_n\} \) in \( X \) does not have a convergent subsequence.

6. Let \( F : \mathbb{R}^3 \to \mathbb{R}^3 \) be given by \( F(x, y, z) = (x^2 - y^3, xy, (z - 1)^4) \). For which points \( p = (x, y, z) \) is \( F \) a diffeomorphism in a neighborhood of \( p \)?

7. Consider the surface \( S = \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 + y^2\} \). Compute the tangent space to \( S \) at \( p = (1, 0, 1) \) and determine the geodesic going from \( p \) to \( q = (0, 0, 0) \) as a parameterized curve.

8. Define the cotangent bundle of a differentiable manifold. (Hint: first define the cotangent space at a point.)

9. Describe all smooth surfaces in \( \mathbb{R}^3 \) with coordinates \((x, y, z)\) such that the pullback of the one-form \( \theta := dy - zdx \) is identically zero.
10. Let \( r > 0 \) be a constant and consider the surface \( S = \{(x, y, z) \in \mathbb{R}^3 \mid r = x^2 + y^2\} \). Compute the Gauss and mean curvature functions on \( S \). What is the group of isometries of \( S \)?