GEOMETRY AND TOPOLOGY QUALIFIER

Instructions. Answer all questions. Write your name and page number in the upper right corner of each page. Start each problem on a new sheet of paper, and use only one side of each sheet.

1. Let $X$ be a compact metric space. Prove that (a) every real valued continuous function on $X$ has a maximum value; (b) every continuous function on $X$ is uniformly continuous.

2. Let $S^1$ be the circle $S^1 = \{e^{2\pi i t} : t \in \mathbb{R}\}$. Define an equivalence relation $\sim$ on $S^1$ by: $e^{2\pi i t_1} \sim e^{2\pi i t_2}$ iff $t_1 - t_2$ is an integer multiple of $\sqrt{2}$. Prove that the quotient space $X = S^1 / \sim$ is not Hausdorff and find all continuous functions on $X$.

3. Let $SL_n(\mathbb{R})$ be the $n \times n$ matrices with real entries and determinant 1. Prove that $SL_n(\mathbb{R})$ is a noncompact smooth manifold and find the dimension of the manifold.

4. Let $X_1 = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$ and $X_2 = -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2}$ be two smooth vector fields on $\mathbb{R}^2$. Is there a coordinate chart $(y, U)$ for a neighborhood $U$ of $(1, 1)$ such that $X_1 = \frac{\partial}{\partial y_1}$ and $X_2 = \frac{\partial}{\partial y_2}$? Prove your claim.

5. Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be given the standard Riemannian metric inherited from $\mathbb{R}^3$. The projective space $\mathbb{P}^2$ is obtained from $S^2$ by identifying all antipodal points, i.e. $\mathbb{P}^2 = S^2 / \sim$, where two points $p \sim q$ iff $p = -q$. Let $\pi$ be the canonical map from $S^2$ to $\mathbb{P}^2$ by mapping each point to its equivalence class.

(a) prove that $\mathbb{P}^2$ is a compact smooth manifold;
(b) prove that there exists a unique Riemannian metric on $\mathbb{P}^2$ such that its pullback (by $\pi$) to $S^2$ is the standard Riemannian metric on $S^2$;
(c) find all geodesics on $\mathbb{P}^2$ with respect to the Riemannian metric in (b) and prove that every two distinct geodesics intersect exactly once;
(d) compute the curvature of the above Riemannian metric on $\mathbb{P}^2$.

Date: August, 2013.