INSTRUCTIONS
• There are 8 problems. Work on all of them.
• Prove your assertions.
• Use separate sheet of paper for each problem and write only on one side of
  the paper.
• Write your name on the top right corner of each page.

Problem 1. Show that a bijection \( f : X \rightarrow Y \) is a homeomorphism if and only
if \( f(A) = f(A) \) for every \( A \subset X \).

Problem 2. Prove that the one-point compactification of the half-open interval
\([0, 1)\) is homeomorphic to the closed interval \([0, 1]\).

Problem 3. a) Give the definition of a connected component of a topological
  space.
  b) Let \( X \) be a topological space, and let \( X' \subset X \). Show that the connected
  component of \( x \in X' \) in the subspace \( X' \) is a subset of the connected component
  of \( x \) in \( X \).

Problem 4. Prove that a metric space \( X \) is compact if and only if every continuous
  function \( f : X \rightarrow \mathbb{R} \) is bounded.

Problem 5. Let \( \mathbb{R}^3 \) have coordinates \((x, y, z)\) and the standard Euclidean struc-
  ture. Let \( S \subset \mathbb{R}^3 \) be the surface parametrized locally by \( x = t + s, y = t^2 + 2ts, 
  z = t^3 + 3st^2 \), where \( s, t > 0 \). Using any method you please, determine the Gauss
  curvature function \( K(s, t) \).

Problem 6. Let \( M \) be a differentiable manifold, and let \( x \in M \).
   (1) Define (without reference to the tangent space), the cotangent space of \( M 
  \) at \( x \), \( T^*_x M \).
   (2) Define (without reference to the cotangent space), the tangent space of \( M 
  \) at \( x \), \( T_x M \).
   (3) Show that, with the above definitions, \( T_x M \) and \( T^*_x M \) are dual vector
  spaces.

Problem 7. Consider the following Lie subgroup of \( \text{GL}_3 \mathbb{R} \):
\[
G := \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} | x, y, z \in \mathbb{R} \right\}
\]
Determine the Lie algebra of \( G \) as a subalgebra of \( \text{gl}_3 \mathbb{R} \).

Problem 8. On \( \mathbb{R}^3 \) with coordinates \((x, y, z)\), let \( \theta = dx + f(z)dy \), for some
function \( f(z) \). State a necessary and sufficient condition on \( f(z) \) such that for each
point of \( \mathbb{R}^3 \) there exists a surface \( S \) whose tangent plane is annihilated by \( \theta \), i.e., if
\( v, w \) is a basis for \( T_p S \), then \( \theta_p(v) = 0 \) and \( \theta_p(w) = 0 \).