Geometry-Topology Qualifying Examination
January 2009

Notation. $\mathbb{R}$ denotes the real numbers, and $\mathbb{R}^n$ denotes Euclidean $n$-dimensional space. Similarly $\mathbb{C}$ denotes the complex numbers, and $\mathbb{C}^n$ denotes complex $n$-dimensional space.

1. Let $M^m$ be a smooth $m$-dimensional manifold and let $N^n$ be a closed embedded $n$-dimensional submanifold of $M$. Define the tangent bundle $T(M)$ as a $2m$-dimensional smooth manifold. Show that $T(N)$ is a closed embedded submanifold of $T(M)$.

2. If $X$ is countably compact and $Y$ is Hausdorff and second countable, then a continuous bijection $f : X \to Y$ is a homeomorphism. (Note: $X$ is countably compact if every countable cover has a finite subcover.)

3. Denote $I = [0,1]$. Let the space $X$ be the set $I \times I$ with the lexicographic order topology ($(a,b) < (c,d)$ if either $a < c$, or $a = c$ and $b < d$). Prove that $X$ is first countable and compact, but not separable.

4. Let $X$ be a paracompact Hausdorff space. Show that if $X$ contains a dense, Lindeløf subspace $S$, then $X$ is also Lindeløf.

5. Let $G$ be a topological group and let $H$ be a subgroup of $G$, and denote by $G/H$ the set of left cosets of $H$ in $G$. Show that $\pi : G \to G/H$ is an open map.

6. Prove that $M := \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^4 - z^3 = 1\}$ is a submanifold of $\mathbb{R}^3$.

7. Classify the minimal surfaces $S \subset \mathbb{R}^3$ with zero Gauss curvature.

8. Let $M$ be a manifold, and $N \subset M$ a submanifold. Suppose that $X$ and $Y$ are smooth vector fields on $M$ with the property that $X_p, Y_p \in T_p N$ for all $p \in N$. Prove that $[X,Y]_p \in T_p N$.

9. Let $X$ and $Y$ be two vector fields on $\mathbb{R}^3$ with the property that $X_p$ and $Y_p$ are linearly independent for all $p \in \mathbb{R}^3$. Pick a 1-form $\omega$ on $\mathbb{R}^3$ with the property that $\omega(X) = 0 = \omega(Y)$. Prove that though every point $p \in \mathbb{R}^3$ there exists a surface $S$ such that the tangent spaces $T_q S$, $q \in S$, are spanned by $X_q$ and $Y_q$ if and only if $\omega \wedge d\omega = 0$.

10. Prove that the saddle surface $z = xy$ is ruled. Compute its Gauss curvature.