Solutions to 2011 Power Team Exam

Problem 1. Let $m \geq n \geq k \geq 2$ be natural numbers. A set M of m points with the property (m, n, k) is given in the plane. Show that the minimal number of disks of radius 1 (diameter 2) that is always sufficient to cover all the points in M is $n - k + 1$. What can you say if $k = 1$?

We first point out that there are at most $n-k+1$ points such that the distance from a point to its nearest neighbor is more than 1. Suppose there are $n - k + 2$ such points. Add $k - 2$ points to the set. Then, since M has property (m, n, k), k of these points can be covered by a disk of diameter 1. Since at least 2 of these points must come from the original set of $n - k + 2$ points, this contradicts the fact that the distance between these two points must be greater than 1. Thus, we cannot have $n - k + 2$ points whose distance from one point to any other is greater than 1.

To see that $n - k + 1$ disks of radius 1 is sufficient to cover any set, M, with property (m, n, k), pick any point A_1 of M, cover it with a disk, D_1, of radius 1. If this disk covers M, we are done. If not, pick a point A_2 not in D_1 cover it with a disk, D_2 of radius 1. Note that the distance between A_1 and A_2 is greater than 1. Continue in this fashion. After $n - k + 1$ disks are chosen, the set M must be covered, for other wise we can find a point in M, which is further than 1 unit from the first $n-k+1$ points found, which are all at a distance greater than 1 from each other.

Lastly we need to exhibit a set with property (m, n, k) which cannot be covered with fewer than $n-k+1$ disks of radius 1. Put $m-(n-k)$ points inside a disk of diameter 1. Then place $n - k$ points so that the distance of each of them from the disk and from each other is greater than 2. Any collection of n points must have at least k of them inside the disk. Thus, our set has property (m, n, k), and it needs $n - k + 1$ disks of diameter 2 to cover the entire set.

If $k = 1$, then the minimal number of disks if m. An example of a set with property $(m, n, 1)$, that needs m disks of diameter 2 to cover it is given by $M = \{(10i, 0)\}_{i=0}^{m-1}$.

Problem 2. Calculate the following numbers (be sure to carefully justify your answer in each case).

(2a) $\#(4, 2, 1) = 4$

Take any set of 4 points for which no point is closer than 2 to its nearest neighbor. In fact it is clear that no matter what the values of m and n are, we have

$$\#(m, n, 1) = m$$

(2b)$\#(4, 2, 2) = 2$

Take an equilateral triangle with edge length equal to 1, and put a forth point inside the triangle. This set has property $(4, 2, 2)$ and needs 2 disks to cover it. To see that 2 disks suffice for a set with this property. Pick any 2 points from the set. They can be covered by a disk of diameter 1. There are at most 2 points not in this disk, and they can be covered by a second disk.

(2c) $\#(4, 3, 1) = 4$. See the answer to part (2a).

(2d) $\#(4, 3, 2) = 3$.

To see that 3 disks are sufficient to cover a set with property $(4, 3, 2)$, note that 2 points can be covered by a single disk, which leaves at most two points not covered. Since each of these points can be covered by a single disk, we will have used at most 3 disks to cover the set.

To see that 3 disks are needed, let M be the set consisting of the vertices of an equilateral triangle with edge length 1. Put a fourth point more than 1 unit from any of the vertices. See the figure below.

This set has property $(4, 3, 2)$, for no matter what subset of 3 points is picked at least two will be triangular vertices, and any two vertices are distance 1 from each other. Since we need 2 disks to cover the triangle, and the fourth point cannot be in either of these disks, 3 disks are needed to cover the set.

Problem 3. Calculate the following numbers (be sure to carefully justify your answer in each case).

(3a) $\#(5,4,2)$=4.

To see that 4 disks are sufficient we note that when we cover 2 points with one disk, there will be at most 3 points uncovered, and each of these can be covered with a single disk. To see that 4 disks may be required let M be a set with 5 points such that three points are the vertices of an equilateral triangle, with edge length 1. The last 2 points are placed far away from the triangle and each other. This set has property $(5,4,2)$, since any subset with 4 points must contain at least 2 of the triangular vertices, which can be covered with a single disk, and it will take 4 disks to cover the set M.

(3b) $\#(5,4,3) = 2$.

Problem 1. tells us that at least two disks are required, for if there is a set that requires 2 disks of radius 1 to cover it, then at least 2 disks of radius 1/2 are also needed.

To see that two disks of diameter 1 are sufficient let $M = \{A,B,C,X,Y\}$ be a set with property $(5,4,3)$ such that the points $A, B,$ and C, can be covered by one disk, and the set M cannot be covered with 2 disks. Note this means that the points X and Y cannot be covered with a single disk.

Consider the following subsets of M:
$\{A,B,X,Y\}$, $\{A,C,X,Y\}$, and $\{B,C,X,Y\}$. Each of these has a subset of three points, which can be covered by a single disk, and each of these subsets must contain only one of the points X or Y, since by supposition these two points cannot be covered by a single disk. Denote these subsets by T_A, T_B, T_C, where $T_A = \{B,C,X\}$ or $\{B,C,Y\}$, etc. If the points X and Y both occur in the sets T_A, T_B, T_C, say $X \in T_A$ and $Y \in T_B$ then we have

$$T_A \cup T_B = \{B,C,X\} \cup \{A,C,Y\} = \{A,B,C,X,Y\} = M,$$

and M can be covered by two disks. Thus, only one of the points X or Y can be in the sets T_A, T_B, T_C. Suppose it is X.

Then the four sets

$$\{A, B, X\},\ \{A, C, X\},\ \{B, C, X\},\ \{A, B, C\}$$

can all be covered by a single disk. Thus, the set $\{A, B, C, X\}$ has property $(4, 3, 3)$, and by Problem 6. can be covered by a single disk, which means that the original set M can be covered by a single disk. Thus, the supposition that there is such a set M leads to a contradiction. Hence we see that $\#(5, 4, 3) = 2$.

(3c) $\#(5, 4, 4) = 1$. This follows from the fact that a set with property $(5, 4, 4)$ also has property $(5, 3, 3)$, and for such sets we know that the minimal number of disks needed is 1.

Problem 4. Calculate the following numbers (be sure to carefully justify your answer in each case).

(4a) $\#(6, 5, 2) = 5$.

Since 2 points can be covered with one disk leaving at most 4 points uncovered, we see that 5 disks will suffice to cover the 6 points. A set with property $(6, 5, 2)$ that needs 5 disks to cover it is given by placing 3 points at the vertices of an equilateral triangle with edge length 1 and the other three points such that they all have distance greater than 1 from each other as well as the first three points. Any subset of 5 points must contain 2 of the triangular vertices, and they can be covered by a disk of diameter 1.

(4b) $\#(6,5,3)=3$

From Problem 1 $\#(6,5,3) \geq 5-3+1=3$. It remains to be shown that 3 disks are always sufficient.

At least 3 of the 6 points can be covered by a disk of diameter 1. Let A, B and C be three such points and denote the other 3 points by X, Y and Z.

If any 2 of the points X, Y and Z are within distance 1 from each other than these two points can be covered by a disk of diameter 1 and 3 disks are sufficient to cover all 6 points $(3+2+1=6)$.

Assume that the mutual distances between X, Y and Z are greater than 1. Thus no disk of diameter 1 contains 2 of these three points. Consider the three sets of points
$S_A=\{B,C,X,Y,Z\}$, $S_B=\{A,C,X,Y,Z\}$ and $S_C=\{A,B,X,Y,Z\}$. By the property $(6,5,3)$ each contains a subset of three points, denoted T_A, T_B and T_C respectively, that can be covered by a disk of diameter 1. Since no disk of diameter 1 contains two of the points X, Y and Z the subsets T_A, T_B and T_C always involve exactly two of the points A, B and C. If the union of any two of the sets T_A, T_B and T_C contains 5 points we again see that 3 disks of diameter 1 suffice $(5+1=6)$. Otherwise all three sets T_A, T_B and T_C contain the same point among X, Y and Z. Assume, without loss of generality, that X is contained in T_A, T_B and T_C, Then the points A, B, C and X satisfy the property $(4,3,3)$ and, by Problem 6, can be covered by a single disk of diameter 1. Since the other 2 points can be covered by 2 disks, 3 disks suffice in each case.

(4c) $\#(6,5,4) = 2$

Let M be a set with property $(6,5,4)$. We know that 4 of the 6 points can be covered by a single disk of diameter 1. Label these points A, B, C, D. Label the other 2 points X and Y. We can assume that the distance between X and Y is greater than 1 for other wise they can be covered by a disk of diameter 1, which means we can cover M with 2 disks. Consider the 4 sets: $S_A = \{B, C, D, X, Y\}, S_B = \{A, C, D, X, Y\}, S_C = \{A, B, D, X, Y\}$, and $S_D = \{A, B, C, X, Y\}$. Since each of these sets has 5 points, and M has property $(6,5,4)$, each of these sets has a subset of 4 points that can be covered by a single disk. Moreover each these subsets must contain exactly one of X and Y. If the union of two of these subsets contains both X and Y, then that union is all of M, and we have covered M with 2 disks. So suppose now that only the point X is in these 4 subsets. This means that the sets

$$\{B, C, D, X\}, \{A, C, D, X\}, \{A, B, D, X\}, \text{ and } \{A, B, C, X\}$$

can each be covered by a single disk. That is, the set $\{A, B, C, D, X\}$ has property $(5,4,4)$. Such a set also has property $(5,3,3)$, and thus, can be covered by a single disk. Which means that the original set M can be covered by 2 disks.

To see that 2 disks are necessary, use Problem 1.

Problem 5. Four disks of diameter 1 are placed in the plane. If every proper subcollection of the four disks have a common point, then all four disks have a common point.

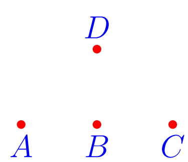
Let d_A, d_B, d_C and d_D be four disks of diameter 1 in the plane such that any three of them have a common point. Let A be a point in $d_B \cap d_C \cap d_D$, B a point in $d_A \cap d_C \cap d_D$, C a point in $d_A \cap d_B \cap d_D$ and D a point in $d_A \cap c_B \cap d_C$.

There are 4 possibilities for the arrangement of the four points: all 4 lie on a straight line, only 3 lie on a straight line, no 3 points lie on a single straight line. This last case has two parts, three points form a triangle with the fourth point outside the triangle, or the fourth point inside the triangle.

4 points on a line Assume that the points lie on the line in increasing alpha order.

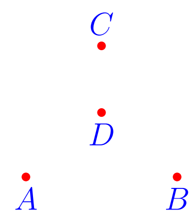
The only disk of the four, which may not contain the point C is d_C. However, that disk contains the points A and D, and must also contain the point C. Hence, the point C is in all 4 disks.

3 points on a line

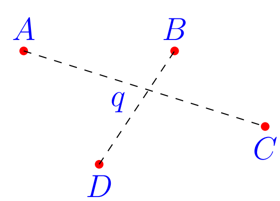


The only disk which may possibly not contain the point B is d_B, but this disk contains the points A and C, and hence must contain the point B. Thus, all 4 disks contain the point B

Triangle. Assume one of the four points A, B C and D is inside the triangle (including the degenerate cases) determined by the other three points. Without loss of generality assume that D is inside the triangle ABC. Since A, B and C all belong to d_D the whole triangle ABC, along with the interior point D is contained in d_D. Thus, the point D is contained in all 4 disks.



Convex quadrilateral. The four points A, B, C and D form a convex quadrilateral Q. Without loss of generality assume that AC and BD are the diagonals of Q. Since both A and C belong to d_B and d_D, the whole diagonal AC belongs to $d_B \cap d_D$. Similarly, the whole diagonal BD belongs to $d_A \cap d_C$. But then the intersection of the two diagonals, q, belongs to all four disks d_A, d_B, d_C and d_D.



Problem 6. Four points are placed in the plane. If every proper subset of these four points can be covered by a disk of diameter 1, then all four points can be covered by such a disk.

Label the four points A, B, C, D. Let d_X denote the disk of radius 1/2 centered at the point X. Since every proper subset of these four points can be covered by a single disk of diameter one, there are points A', B', C', D' such that

$$\{A, B, C\} \subset d_{D'},\ \{A, B, D\} \subset d_{C'},\ \{A, C, D\} \subset d_{B'},\ \{B, C, D\} \subset d_{A'}$$

Note that if $X \in d_Y$ then $Y \in d_X$. This is nothing more then saying if X is within 1/2 of Y, then Y is within 1/2 of X. Thus, we see that we have the following:

$$\{A', B', C'\} \subset d_D,\ \{A', B', D'\} \subset d_C,$$

$$\{A', C', D'\} \subset d_B,\ \{B', C', D'\} \subset d_A$$

These four disks, d_A, d_B, d_C, and d_D have the property that the intersection of any three of them has a common point. For example $A' \in d_B \cap d_C \cap d_D$. Thus, by problem 5, the intersection of all 4 of these disks contains at least one point. Call this point P. That is, $P \in d_A \cap d_B \cap d_C \cap d_D$. This then implies that $\{A, B, C, D\} \subset d_P$.