Thirteenth Annual Texas A&M University
High School Mathematics Contest
Power Team Solutions

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Question 1 *Show that ϕ is one-to-one. That is, show that if $\phi(s) = \phi(t)$, then $s = t$.*

Solution:

Let $\phi(s) = (x_s, y_s)$ and $\phi(t) = (x_t, y_t)$. If $\phi(s) = \phi(t)$, then

$$\begin{aligned} y_s &= y_t \\ \frac{s^2-1}{s^2+1} &= \frac{t^2-1}{t^2+1} \\ \frac{-2}{s^2+1} + 1 &= \frac{-2}{t^2+1} + 1 \\ s^2+1 &= t^2+1. \end{aligned}$$

As $x_s = x_t \Rightarrow \dfrac{2s}{s^2+1} = \dfrac{2t}{t^2+1}$ and $s^2+1 = t^2+1$, we have $s = t$.

Question 2 *Show that the range of* ϕ *is the unit circle minus the point* $(0,1)$*. That is, show for any* t *that* $\phi(t)$ *is a point on the unit circle and that this point is not* $(0,1)$*. Then show that for any point* (x,y)*, which satisfies* $x^2+y^2=1$ *with* $y \neq 1$ *there is a real number* t *such that*

$$\phi(t) = (x,y).$$

Be sure to find a formula for t *in terms of* x *and* y*.*

Solution:

First, for $\phi(t) = (x,y)$,

$$x^2+y^2 = \left(\frac{2t}{t^2+1}\right)^2 + \left(\frac{t^2-1}{t^2+1}\right)^2 = \frac{4t^2}{t^4+2t^2+1} + \frac{t^4-2t^2+1}{t^4+2t^2+1} = 1.$$

Thus $\phi(t)$ is on the unit circle. Also, $y \neq 1$ as if $\frac{t^2-1}{t^2+1} = 1$, then $t^2-1 = t^2+1$, a contradiction.

Next, consider any point (x,y) on the unit circle $x^2+y^2=1$ with $y \neq 1$. Let $t = \frac{x}{1-y}$ (which is well defined as $y \neq 1$). Then

$$t^2+1 = \frac{x^2}{1-2y+y^2} + 1 = \frac{1-2y+x^2+y^2}{1-2y+y^2} = \frac{2-2y}{1-2y+y^2} = \frac{2}{1-y}.$$

Thus,

$$\begin{aligned}
\phi(t) &= \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right) \\
&= \left(\frac{2\frac{x}{1-y}}{\frac{2}{1-y}}, \frac{-2}{t^2+1} + 1\right) \\
&= \left(\frac{2x}{2}, \frac{-2}{\frac{2}{1-y}} + 1\right) \\
&= (x, (y-1)+1) \\
&= (x,y).
\end{aligned}$$

We might note that as both ϕ and ϕ^{-1} are continuous, we have shown that $\mathbb{R}$ is homeomorphic to the circle with a point removed.

Question 3 *Show that* ϕ *maps the negative real axis onto the left side of the circle* $(x < 0)$*, the positive real axis onto the right side of the circle* $(x > 0)$*, and if* $-1 < t < 1$*, then* $\phi(t)$ *lies on the bottom half of the circle* $(y < 0)$*.*

Solution:

We have $x = \frac{2t}{t^2+1}$. As $t^2 + 1 \geq 1$ for all t, x has the same sign as t. Thus, the negative real axis maps onto the left side of the circle, and the positive real axis onto the right side. Zero maps to $(0, -1)$.

If $|t| < 1$, then $t^2 < 1$ and $t^2 - 1 < 0$. As $t^2 + 1 > 0$, $y = \frac{t^2-1}{t^2+1} < 0$.

Question 4 *To picture the mapping ϕ, begin with a point t on the x-axis. Draw the line segment from $(t, 0)$ to $(0, 1)$. This line segment intersects the circle at a second point (x, y). Show that $x = \frac{2t}{t^2+1}$ and $y = \frac{t^2-1}{t^2+1}$. That is, the point (x, y) is $\phi(t)$.*

Solution:

The equation of the line segment from $(t, 0)$ to $(0, 1)$ is $y = 1 - \frac{x}{t}$. Plugging this into the equation $x^2 + y^2 = 1$ of the circle yields

$$\begin{aligned} x^2 + \left(1 - \frac{x}{t}\right)^2 &= 1 \\ x^2 + 1 - \frac{2x}{t} + \frac{x^2}{t^2} &= 1 \\ x^2 - \frac{2x}{t} + \frac{x^2}{t^2} &= 0 \\ xt^2 - 2t + x &= 0 \\ x &= \frac{2t}{t^2+1}. \end{aligned}$$

Plugging this into $y = 1 - \frac{x}{t}$ yields

$$y = 1 - \frac{2}{t^2+1} = \frac{t^2-1}{t^2+1}.$$

Question 5 *Prove that for real numbers t and s, (a_t, a_s) is a vertical pair if and only if $st = 1$.*

Solution:

We first show that if (a_t, a_s) is a vertical pair, then $st = 1$. If $s = t = 1$ or $s = t = -1$, then clearly $st = 1$. If a_t and a_s are distinct points on the same vertical line, then a_t and a_s have the same x-coordinate but different y-coordinates. Thus,

$$\frac{2t}{t^2+1} = \frac{2s}{s^2+1} \iff s^2t + t = st^2 + s \iff st(s-t) = s - t.$$

As $a_t \neq a_s$, $t \neq s$ and we can divide by $s - t$ to obtain $st = 1$.

The other direction is simply the reverse. If $st = 1$, then either $s = t = 1$, $s = t = -1$, or $s \neq t$. In the first two cases, (a_t, a_s) is a vertical pair by definition. Otherwise,

$$st(s-t) = s - t \iff s^2t + t = st^2 + s \iff \frac{2t}{t^2+1} = \frac{2s}{s^2+1}.$$

(We can divide by $t^2 + 1$ and $s^2 + 1$ above as both are $\neq 0$.) Thus, a_t and a_s have the same x-coordinate and lie on the same vertical line, but are distinct as $t \neq s$ and ϕ is injective.

Question 6 *Prove that for real numbers* t *and* s, (a_t, a_s) *is a horizontal pair if and only if* $t = -s$.

Solution:

We first show that if (a_t, a_s) is a horizontal pair, then $t = -s$. If $s = t = 0$, then clearly $t = -s$. Otherwise, a_t and a_s are distinct points on the same horizontal line, and thus have the same y-coordinate. Thus,

$$\frac{t^2-1}{t^2+1} = \frac{s^2-1}{s^2+1} \iff \frac{-2}{t^2+1}+1 = \frac{-2}{s^2+1}+1 \iff t^2+1 = s^2+1 \iff t = \pm s.$$

As $a_t \neq a_s$, $t \neq s$ and $t = -s$.

For the other direction, if $t = -s$, then either $s = t = 0$ or $s \neq t$. If $s = t = 0$ then (a_t, a_s) is a horizontal pair by definition. Otherwise, $t^2 = s^2$ and $\frac{t^2-1}{t^2+1} = \frac{s^2-1}{s^2+1}$, implying that a_t and a_s have the same y-coordinate and lie on the same horizontal line, but are distinct as $t \neq s$ and ϕ is injective.

Question 7 *If* (a_t, a_s) *is an antipodal pair, but not a vertical pair, how are* s *and* t *related?*

Solution:

Reflect a_t across the y-axis to a_r, as shown, forming the horizontal pair (a_t, a_r). We know that such an r exists because ϕ is a bijection. Reflecting a_r across the x-axis yields a_s. Thus, (a_r, a_s) is a vertical pair. Thus, $r = -t$ and $rs = 1$. Thus, $st = -1$.

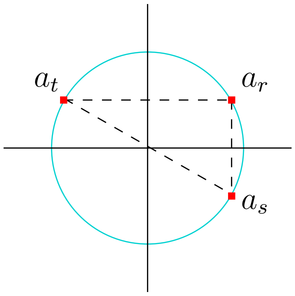


Figure 1: Three points forming vertical, horizontal, and antipodal pairs.

Question 8 *Suppose* (a_t, a_s) *is not a vertical pair. Then the straight line through them (if* $a_t = a_s$ *use the tangent line to the circle at that point) intersects the* y*-axis at the point* $(0, y)$*. Find* y *in terms of* t *and* s*, and simplify your answer.*

Solution:

We have

$$a_t = \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right), \qquad a_s = \left(\frac{2s}{s^2+1}, \frac{s^2-1}{s^2+1}\right).$$

The slope of the line between a_s and a_t is

$$\frac{\frac{t^2-1}{t^2+1} - \frac{s^2-1}{s^2+1}}{\frac{2t}{t^2+1} - \frac{2s}{s^2+1}} = \frac{\frac{-2}{t^2+1} + 1 - \frac{-2}{s^2+1} - 1}{\frac{2t}{t^2+1} - \frac{2s}{s^2+1}} = \frac{\frac{1}{s^2+1} - \frac{1}{t^2+1}}{\frac{t}{t^2+1} - \frac{s}{s^2+1}}$$
$$= \frac{t^2-s^2}{t\left(s^2+1\right) - s\left(t^2+1\right)} = \frac{t^2-s^2}{(t-s) - st(t-s)} = \frac{s+t}{1-st}.$$

(If $a_t = a_s$ then, by continuity, this will be the slope of the tangent line.) Thus, the equation of the line between s and t is

$$y = \frac{s+t}{1-st}\left(x - \frac{2s}{s^2+1}\right) + \frac{s^2-1}{s^2+1}.$$

The y-intercept of this line is thus

$$\begin{aligned}
&\frac{s+t}{1-st}\left(-\frac{2s}{s^2+1}\right) + \frac{s^2-1}{s^2+1} \\
&= \frac{2s^2 + 2st + (st-1)\left(s^2-1\right)}{(st-1)\left(s^2+1\right)} \\
&= \frac{s^3t + s^2 + st + 1}{s^3t - s^2 + st - 1} \\
&= \frac{2s^2+2}{(st-1)\left(s^2+1\right)} + 1 \\
&= \frac{2}{st-1} + 1 \\
&= \frac{st+1}{st-1}.
\end{aligned}$$

Question 9 *Draw the straight line through the point* $(0, y)$ *and the point* $(1, 0)$*, where* $(0, y)$ *refers to the point found in the previous problem. Let* a_u *denote the second point of intersection of the line with the circle. Prove that* $u = st$*.*

Solution:

Note that $(1, 0) = a_1$. By the previous problem, $y = \frac{st+1}{st-1}$. Note, however, that as $(0, y)$ is also the intersection of the line through a_u and a_1 with the y-axis, the previous problem also implies that $y = \frac{1 \cdot u+1}{1 \cdot u-1} = \frac{u+1}{u-1}$. Thus,

$$\begin{aligned} \frac{st+1}{st-1} &= \frac{u+1}{u-1} \\ st \cdot u - st + u - 1 &= st \cdot u + st - u - 1 \\ 2u &= 2st \\ u &= st. \end{aligned}$$

Question 10 *Suppose that (a_t, a_s) is not a horizontal pair. Then the line through a_t and a_s (if $s = t$ then use the tangent line to the circle at the point a_t) intersects the horizontal line $y = 1$ at the point $(x, 1)$. Find x in terms of t and s, and simplify your answer.*

Solution:

In our solution to Part 8, we calculated that the equation of the line between a_s and a_t (or the tangent line, if $a_s = a_t$) is

$$y = \frac{s+t}{1-st}\left(x - \frac{2s}{s^2+1}\right) + \frac{s^2-1}{s^2+1}.$$

Plugging in $y = 1$ and solving for x yields

$$\begin{aligned} 1 &= \frac{s+t}{1-st}\left(x - \frac{2s}{s^2+1}\right) + \frac{s^2-1}{s^2+1}, \\ x &= \frac{2s}{s^2+1} + \left(1 - \frac{s^2-1}{s^2+1}\right)\left(\frac{1-st}{s+t}\right) \\ &= \frac{2s}{s^2+1} + \left(\frac{2}{s^2+1}\right)\left(\frac{1-st}{s+t}\right) \\ &= \frac{2s(s+t)+2(1-st)}{(s^2+1)(s+t)} \\ &= \frac{2s^2+2}{(s^2+1)(s+t)} \\ &= \frac{2}{s+t}. \end{aligned}$$

Question 11 *Draw the straight line through the point $(x, 1)$ found in the previous problem and the point $(0, -1)$. This line intersects the circle in a second point, call it a_u. Show that $u = s + t$.*

Solution:

Note that $(0, -1) = a_0$. By the previous problem, $x = \frac{2}{s+t}$. Note, however, that as $(x, 1)$ is also the intersection of the line through a_u and a_0 with the line $y = 1$, the previous problem also implies that $x = \frac{2}{u+0} = \frac{2}{u}$. Thus, $\frac{2}{u} = \frac{2}{s+t}$, and $u = s + t$.

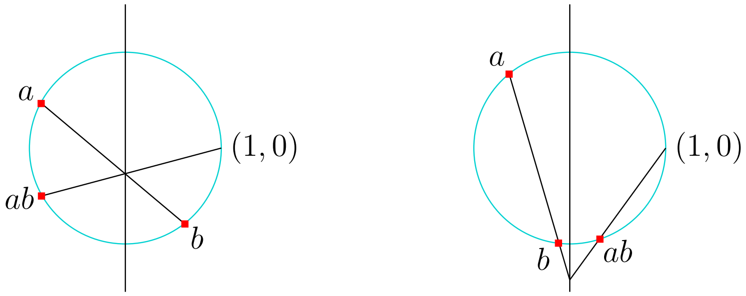
Question 12 *Using this geometrical way of multiplying numbers show that the product of a negative and a positive is negative, and that the product of two negative numbers is positive.*

Solution:

As shown in Part 3, points on the circle representing negative numbers are on the left side of the y-axis, and those representing positive number are on the right side.

If we are multiplying a negative number a and a positive number b, as shown in Figure 2(a), a and b are on different sides of the y-axis, and thus the line between them intersects the y-axis inside the circle. The line between $(1, 0)$, on the right side of the circle, and the intersection point thus intersects the circle again on the left side of the y-axis, yielding a negative product ab.

If we are multiplying two negative numbers a and b, as shown in Figure 2(b), a and b are on the same side of the y-axis, and thus the line between them does not intersect the y-axis on the segment between them. Instead, the intersection is outside of the circle. The line between the intersection point and $(1, 0)$ thus intersects the circle to the right of the y-axis, yielding a positive product ab.



(a) Multiplying a negative number a and a positive number b.

(b) Multiplying two negative numbers a and b.

Figure 2: Multiplying numbers of particular signs.

Question 13 *Show that* (x, y) *is a rational point on the unit circle not equal to* $(0, 1)$ *if and only if there is a rational number* t *such that* $\phi(t) = (x, y)$. *Deduce from this that there are infinitely many rational points on the unit circle.*

Solution:

First, if there is a rational number t such that $\phi(t) = (x, y)$, then

$$x = \frac{2t}{t^2+1} \qquad \text{and} \qquad y = \frac{t^2-1}{t^2+1},$$

both of which are rational.

For the other direction, recall that we showed in Part 2 that if $t = \frac{x}{1-y}$, then $\phi(t) = (x, y)$. As x and y are rational, and $y \neq 1$, t is clearly rational.

Lastly, as there are infinitely many rational t, and as for each rational t, $\phi(t)$ is a distinct rational point, there are infinitely many rational points on the unit circle.

Question 14 *Explain why there are infinitely many values of* θ *for which* $\cos\theta$*,* $\sin\theta$*,* $\tan\theta$*,* $\sec\theta$*,* $\csc\theta$*, and* $\cot\theta$ *are all rational numbers.*

Solution:

For a point (x, y) on the unit circle centered at the origin, $x = \cos\theta$ and $y = \sin\theta$, where θ is the angle the ray from $(0, 0)$ to (x, y) makes with the positive x-axis. If (x, y) is a rational point, then $\cos\theta = x$, $\sin\theta = y$, $\tan\theta = y/x$, $\sec\theta = 1/x$, $\csc\theta = 1/y$, and $\cot\theta = x/y$ are all rational. As there are, by Part 13, infinitely many rational points on the unit circle, there are infinitely many such θ.

Question 15 *A right triangle is said to be a primitive right triangle if its sides have integer lengths with no common divisors. Thus, the right triangle with side lengths* 3*,* 4*, and* 5 *is a primitive right triangle, but the right triangle with lengths* 6*,* 8*, and* 10 *is not. Show that there are infinitely many primitive right triangles.*

Solution:

Consider the triangle with sides of length r^2+s^2, r^2-s^2, and $2rs$, where r and s are positive integers, $r > s$, $\gcd(r,s) = 1$, and exactly one of r and s is even. This is a right triangle as

$$\left(r^2-s^2\right)^2 + (2rs)^2 = r^4 - 2r^2s^2 + s^4 + 4r^2s^2 = r^4 + 2r^2s^2 + s^4 = \left(r^2+s^2\right)^2.$$

We claim that this right triangle is primitive. Consider $\gcd\left(r^2+s^2, r^2-s^2\right)$. $\gcd\left(r^2+s^2, r^2-s^2\right) = \gcd\left(r^2+s^2, r^2+s^2-\left(r^2-s^2\right)\right) = \gcd\left(r^2+s^2, 2s^2\right)$. As exactly one of r and s is even, r^2+s^2 is odd. Thus, $\gcd\left(r^2+s^2, 2s^2\right) = \gcd\left(r^2+s^2, s^2\right) = \gcd\left(r^2, s^2\right) = 1$. Thus our right triangle is primitive. As we can choose infinitely many r and s satisfying the required conditions, we can construct infinitely many primitive right triangles.

Question 16 *Show that the circle* $x^2 + y^2 = 3$ *contains no rational points.*

Solution:

Suppose that there do exist $x, y \in \mathbb{Q}$ such that $x^2 + y^2 = 3$. Let $x^2 = x_1^2/x_2^2$ and $y^2 = y_1^2/y_2^2$. We rewrite our equation as $a^2 + b^2 = 3k^2$, where $a = x_1^2 y_2^2$, $b = y_1^2 x_2^2$, and $k = x_2^2 y_2^2$ are integers. Thus, $a^2 + b^2 \equiv 0 \pmod 3$. Perfect squares are 0 or 1 modulo 3 ($0^2 \equiv 0$, $1^2 \equiv 1$, $2^2 \equiv 1$). As, $1 + 1 < 3$, we must have $a \equiv b \equiv 0 \pmod 3$. This implies that $9 \mid a^2 + b^2$. However, $9 \nmid 3k^2$, a contradiction.

Question 17 *Find some condition(s) on* r *that will determine whether or not the circle* $x^2 + y^2 = r^2$ *has rational points.*

Solution:

First, if x and y are to be rational, then clearly r^2 must be rational as well. Thus, let $r^2 = kz^2$, where z is rational and k is a squarefree integer. We claim that the circle has rational points if and only if k is not divisible by any prime $p \equiv 3 \pmod 4$.

Suppose that $x^2 + y^2 = kz^2$ for rational x, y, z with k a squarefree integer. Further suppose that $p \mid k$ for some prime $p \equiv 3 \pmod 4$. Clear the denominators and common factors by multiplying by a square to obtain $a^2 + b^2 = kc^2$, for some relatively prime integers a, b, c in the same ratios as x, y, z. Note that if $p \mid a$, then $p \mid a^2$, and as $p \mid k$, we must have $p \mid b^2$. In this case, as a^2 and b^2 are squares, $p^2 \mid a^2 + b^2$, and as k is squarefree, $p \mid c^2$, contradicting the assumption that a, b, c are relatively prime. Thus, $p \nmid a$ and similarly $p \nmid b$. Consider $a^2 + b^2 = kc^2$ modulo p, obtaining $a^2 \equiv -b^2 \pmod p$. The multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times$ is cyclic of order $p - 1$. Let ω be a generator of this group. As $p \nmid a$ and $p \nmid b$, let $a \equiv \omega^\alpha$ and $b \equiv \omega^\beta$. As $(-1)^2 \equiv 1 \pmod p$, $-1 = \omega^{\frac{p-1}{2}}$. Thus, the equation $a^2 \equiv -b^2 \pmod p$ becomes $\omega^{2\alpha} = \omega^{\frac{p-1}{2}}\omega^{2\beta}$. However, this is impossible because the order of the group is even, and $p \equiv 3 \pmod 4$ implies that $\frac{p-1}{2}$ is odd, while 2α and 2β are even. Thus, we cannot have $p \mid k$ for any $p \equiv 3 \pmod 4$.

For the other direction, suppose that our given k is not divisible by any prime $p \equiv 3 \pmod 4$. If $2 \nmid k$, then k is a product of primes congruent to 1 modulo 4, and thus $k \equiv 1 \pmod 4$. As any integer congruent to 1 modulo 4 can be written as a sum of two squares, write $k = a^2 + b^2$ for some integers a and b. Then let $x = az$ and $y = bz$, yielding $x^2 + y^2 = kz^2 = r^2$, as desired. If $2 \mid k$, then, as $4 \nmid k$, we write $k/2 = a^2 + b^2$. Then let $x = (a + b)z$ and $y = (a - b)z$, yielding $x^2 + y^2 = (a^2 + 2ab + b^2)z^2 + (a^2 - 2ab + b^2)z^2 = 2(a^2 + b^2)z^2 = 2(k/2)z^2 = kz^2 = r^2$.