The following sequence of questions will lead to a method of geometrically adding and multiplying real numbers. Along the way you will also discover some results concerning rational points in R^2. The symbols R and R^2 denote the real numbers and points in the Cartesian plane respectively.

Define $\phi : R \to R^2$ by

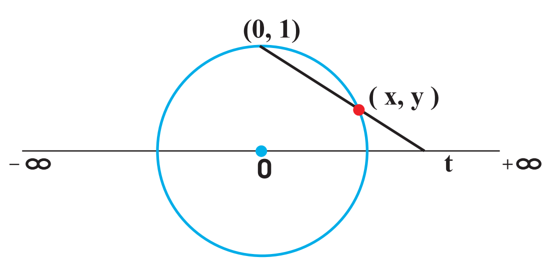
$$\phi(t) = \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right) = (x, y) \ .$$

This function maps the real numbers into the unit circle, and is called a stereographic projection.

1. Show that ϕ is one-to-one. That is, show that if $\phi(s) = \phi(t)$, then $s = t$.
2. Show that the range of ϕ is the unit circle minus the point $(0, 1)$. That is, show for any t that $\phi(t)$ is a point on the unit circle and that this point is not $(0, 1)$. Then show that for any point (x, y), which satisfies $x^2 + y^2 = 1$ with $y \neq 1$ there is a real number t such that

$$\phi(t) = (x, y)\,.$$

   Be sure to find a formula for t in terms of x and y.
3. Show that ϕ maps the negative real axis onto the left side of the circle $(x < 0)$, the positive real axis onto the right half of the circle $(x > 0)$, and if $-1 < t < 1$, then $\phi(t)$ lies on the bottom half of the circle, $(y < 0)$.
4. To picture the mapping ϕ, begin with a point t on the x-axis. Draw the line segment from $(t, 0)$ to $(0, 1)$. This line segment intersects the unit circle at a second point (x, y). Show that $x = \dfrac{2t}{t^2+1}$ and $y = \dfrac{t^2-1}{t^2+1}$. That is, the point (x, y) is $\phi(t)$. See the plot below.



If we associate (0, 1) with $\pm\,\infty$, the unit circle $x^2 + y^2 = 1$ may be regarded as a copy of the extended (adding $\pm\,\infty$) real line.

For any real number t, let a_t denote the point $\left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right)$.

We define the following terms:

A pair $(a_t,\, a_s)$ is a *vertical pair* if one of the following is true

- $s = t = 1$,
- $s = t = -1$,
- $st \neq 0$ and a_t and a_s are two different points on the same vertical line.

A pair $(a_t,\, a_s)$ is a *horizontal pair* if one of the following is true

- $s = t = 0$,
- or a_t and a_s are two different points on the same horizontal line.

A pair $(a_t,\, a_s)$ is an *antipodal pair* if a_t and a_s are the end points of the same diameter of the circle.

5. Prove that for real numbers t and s, $(a_t,\, a_s)$ is a vertical pair if and only if $st = 1$.
6. Prove that for real numbers t and s, $(a_t,\, a_s)$ is a horizontal pair if and only if $t = -s$.
7. If $(a_t,\, a_s)$ is an antipodal pair, but not a vertical pair, how are s and t related? Be sure to prove and simplify your answer.
8. Suppose $(a_t,\, a_s)$ is not a vertical pair. Then the straight line through them (if $a_t = a_s$ use the tangent line to the circle at that point) intersects the y-axis at the point $(0, y)$. Find y in terms of t and s, and simplify your answer. See Figure M below.
9. Draw the straight line through the point $(0, y)$ and the point $(1, 0)$, where $(0, y)$ refers to the point found in the previous problem. Let a_u denote the second point of intersection of the line with the circle. See Figure M below. Prove that $u = st$.

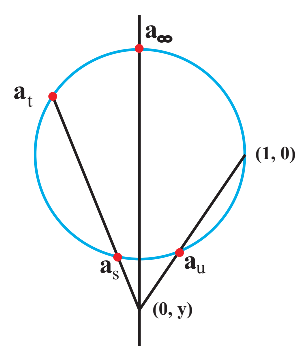


Figure M

10. Suppose that $(a_t,\, a_s)$ is not a horizontal pair. Then the line through a_t and a_s (if $s = t$ then use the tangent line to the circle at the point a_t) intersects the horizontal line $y = 1$ at the point $(x, 1)$. Find x in terms of t and s, and simplify your answer. See Figure A below.

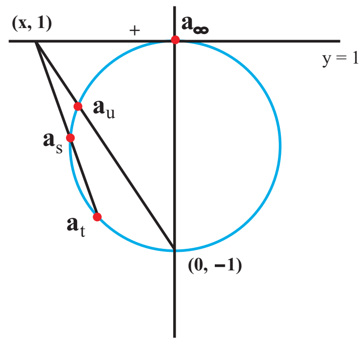


Figure A

11. Draw the straight line through the point $(x, 1)$ found in the previous problem and the point $(0, -1)$. This line intersects the circle in a second point, call it a_u. See Figure A above. Show that $u = s + t$.

We now define addition and multiplication of pairs of numbers
$a = (x, y)$ on the unit circle. For each such pair a and b let

$$a + b = \begin{cases} a_u, \text{ (see problem 11)} & \text{if } a \text{ and } b \text{ are not a horizontal pair} \\ (0, -1), & \text{if } a \text{ and } b \text{ are a horizontal pair} \end{cases}$$

$$ab = \begin{cases} a_u, \text{ (see problem 9)} & \text{if } a \text{ and } b \text{ are not a vertical pair} \\ (1, 0), & \text{if } a \text{ and } b \text{ are a vertical pair} \end{cases} .$$

These two definitions give us a geometric way to add and multiply real numbers. For example, to add the real numbers s and t: compute $\phi(s)$ and $\phi(t)$, add these pairs of numbers together as in problem 11, then find the corresponding u. Now set $s + t = u$. The above results show that whether we compute $s + t$ as usual, or in this geometrical way, we get the same answer. Similar comments apply to computing the product of the real numbers s and t.

Note that vertical pairs correspond to a nonzero number t and its multiplicative inverse $\frac{1}{t}$, and horizontal pairs correspond to a number t and its additive inverse $-t$.

12. Using this geometrical way of multiplying numbers show that the product of a negative and a positive is negative, and that the product of two negative numbers is positive.

A point $(x,\, y) \in R^2$ is called a *rational point* if both x and y are rational numbers. That is, both x and y are ratios of integers.

13. Show that (x, y) is a rational number on the unit circle not equal to $(0, 1)$ if and only if there is a rational number t such that $\phi(t) = (x, y)$. Deduce from this that there are infinitely many rational points on the unit circle.

14. Explain why there are infinitely many values of θ, $0 \leq \theta < 2\pi$, for which $\cos\theta$, $\sin\theta$, $\tan\theta$, $\sec\theta$, $\csc\theta$, and $\cot\theta$ are all rational numbers.

15. A right triangle is said to be a primitive right triangle if its sides have integer lengths with no common divisors. Thus, the right triangle with side lengths 3, 4, and 5 is a primitive right triangle, but the right triangle with lengths 6, 8, and 10 is not. Show that there are infinitely many primitive right triangles.

16. Show that the circle $x^2 + y^2 = 3$ contains no rational points.

17. Find some condition(s) on r that will determine whether or not the circle $x^2 + y^2 = r^2$ has rational points.