

2006 Texas A&M High School Math Contest

Power Team Questions

1. Which is larger $5^{1/3}$ or $\sqrt{2} + 3/10$? (Decimal approximations, no matter how many significant figures, will not be considered as justification.)
2. If m and n are positive integers, show that $\sqrt{2}$ lies between $\frac{m}{n}$ and $\frac{m+2n}{m+n}$.

3. Find the largest interval I of positive real numbers such that

$$\left| \frac{1}{x} - \frac{1}{2} \right| < \epsilon,$$

for every $x \in I$, where ϵ is any fixed positive real number.

4. For each positive integer n , let $U_n = \{(x, y) : |x|^n + |y|^n \leq 1\}$.
 - a. Sketch the region U_n .
 - b. How are the U_n related to each other?
 - c. Determine the set

$$\bigcup_{n=1}^{\infty} U_n,$$

where $\bigcup_{n=1}^{\infty} U_n$ denotes the set consisting of all (x, y) that belong to at least one of the U_n .

5. Let a_k and b_k for $k = 1, 2, \dots, n$ be arbitrary real numbers. Prove the following formula

$$\left(\sum_{k=1}^n a_k b_k \right)^2 = \left(\sum_{k=1}^n a_k^2 \right) \left(\sum_{k=1}^n b_k^2 \right) - \sum_{1 \leq k < j \leq n} (a_k b_j - a_j b_k)^2$$

6. Show that

$$\sum_{k=1}^n |a_k| |b_k| \leq \left(\sum_{k=1}^n a_k^2 \right)^{1/2} \left(\sum_{k=1}^n b_k^2 \right)^{1/2}.$$

7. Given a collection of n positive numbers a_i , $1 \leq i \leq n$ there are three standard "averages" associated with them. They are:

$$H_n = \frac{n}{\frac{1}{a_1} + \dots + \frac{1}{a_n}}$$

$$G_n = (a_1 \cdots a_n)^{1/n}$$

$$A_n = \frac{a_1 + \dots + a_n}{n}$$

Show, for any finite collection of positive numbers a_i , $1 \leq i \leq n$, that the following inequalities are true

$$\min\{a_1, \dots, a_n\} \leq H_n \leq G_n \leq A_n \leq \max\{a_1, \dots, a_n\}.$$

8. A generalization of the $G_n \leq A_n$ inequality is the following: let r_1, \dots, r_n be positive numbers such that $\sum_{i=1}^n r_i = 1$. If $a_i, 1 \leq i \leq n$ are non-negative numbers, show that

$$\left(\prod_{i=1}^n a_i^{r_i} \right) \leq \sum_{i=1}^n r_i a_i,$$

where $\prod_{i=1}^n c_i$ denotes the product of the numbers c_1 through c_n , and $\sum_{i=1}^n c_i$ denotes the sum of the numbers c_1 through c_n . Prove this generalization of the $G_n \leq A_n$ inequality.

9. Suppose Adam buys one dollar's worth of flour each week and Eve buys one pound of flour each week. If the price of flour is not constant from week to week, which one gets the lowest average cost per pound of flour?
10. If $y = f(x)$ is a curve that lies above the x -axis for $a \leq x \leq b$, then the surface area of the surface obtained by rotating the curve about the x -axis is given by the formula

$$\text{Surface Area} = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} dx.$$

Consider the surface S_n obtained by rotating the curve $y = x^n, 0 \leq x \leq 1$, for $n = 1, 2, \dots$ about the x -axis. Determine the limit as n tends to infinity of the surface areas of the surfaces S_n .