# 2008 High School Math Contest
# Solutions to Power Team Exam

A triple of positive integers $(x, y, z)$ is called a Pythagorean Triple if these integers satisfy the equation

$$x^2 + y^2 = z^2 .$$

For example, $(3, 4, 5)$ is a Pythagorean Triple. A Pythagorean Triple is said to be primitive if the integers are relatively prime. That is, they have no common integer factor except 1. The triple $(3, 4, 5)$ is also a primitive Pythagorean Triple.

**1.** If $(x, y, z)$ is a primitive Pythagorean Triple show that

**a.** Exactly one of $x$ and $y$ is odd and hence that $z$ must also be odd,

If both $x$ and $y$ are even that means $z^2$ must be even and thus so is $z$, which means that $x$, $y$, and $z$ are not relatively prime; they're all divisible by 2. If both $x$ and $y$ are odd, then $x^2 + y^2$ must be even. Note that this sum is not divisible by 4. However, since $z^2$ is even $z$ must be even and this forces $z^2$ to be divisible by 4, a contradiction. So the only alternative is that $x$ and $y$ must have opposite parity, which forces $z$ to be odd.

**b.** Exactly one of $x$ and $y$ is divisible by 3,

If both $x$ and $y$ are divisible by 3 then so is $z$, which contradicts the relative primeness of $x$, $y$, and $z$. If neither $x$ nor $y$ is divisible by 3, then $x^2$ and $y^2$ must have remainder 1 when divided by 3. so their sum must have a remainder of 2 when divided by 3, but $z^2$ can only have remainder 0 or 1, when divided by 3. Thus, exactly one of $x$ and $y$ is divisible by 3.

**c.** Exactly one of $x$, $y$, and $z$ is divisible by 5.

Again, since the three integers are relatively prime, not all of $x$, $y$, and $z$ are divisible by 5. If two of them are divisible by 5, then the third must also be divisible by 5, which we know cannot happen. The last case to consider is if none of them are divisible by 5. However, the square of any integer has remainder 1 or 4 when divided by 5: this means that the third must have remainder 0, or 2, or 3 when divided by 5, a contradiction. The only case left is that one in which exactly one of the three is divisible by 5.

**2.** Is there a Pythagorean triple of the form $(2, y, z)$?

The answer is no. Suppose $(2, y, z)$ is a Pythagorean triple. That is,

$$4 + y^2 = z^2$$

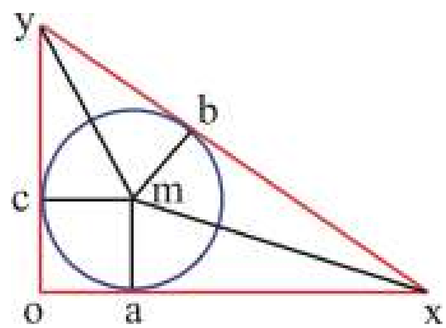
$$4 = (z - y)(z + y) .$$

Since $z - y$ and $z + y$ are integers and are divisors of 4 they can take on the values of 1, 2, or 4 only. If $z - y$ is 1, then $z + y = 4$. This means that $z = 5/2$, a contradiction. If both factors are 2, we then have $z = 2$ and $y = 0$, but Pythagorean triples consist of positive integers. Thus, there is no Pythagorean triple of the form $(2, y, z)$.

**3.** If $(x, y, z)$ is a Pythagorean triple then the isosceles triangles with sides of length $2x$, $z$, $z$ and $2y$, $z$, $z$ have the same area.

The isosceles triangle with sides of length $2x$, $z$, and $z$ has area $xy$ as does the isosceles triangle with sides of length $2y$, $z$, and $z$. The altitude of the first triangle is $y$, while the altitude of the second triangle is $x$.

**4**. Let $(x, y, z)$ be a Pythagorean triple. Let $r$ be the radius of the inscribed circle of the right triangle with sides $x$, $y$, and $z$.




**a**. Show that $r$ must be an integer with the value

$$r = \frac{xy}{x+y+z}$$

In the figure above we have $oc = oa = mc = ma = mb = r$, triangles $mcy$ $myb$ are congruent as are triangles $mbx$ and $axm$. Thus,

$$y - r = yc = yb = z - bx$$

$$x - r = xa = xb$$

Thus, we have

$$\begin{aligned} y - r &= z - bx = z - (x - r) \\ r &= \frac{x+y-z}{2} \\ &= \frac{x+y-z}{2}\,\frac{x+y+z}{x+y+z} \\ &= \frac{xy}{x+y+z}\,. \end{aligned}$$

Since two of the three terms in the numerator of $\frac{x+y-z}{2}$ are odd, the equation $r = \frac{x+y-z}{2}$ shows that $r$ is indeed an integer.

**b**. Let $n$ be any positive integer. Show that there is a right triangle whose sides have integer length and whose inscribed circle has radius $n$.

From problem 6 below we see that if $k$ is any odd integer greater than 1, then $\left(k, (k^2-1)/2, (k^2+1)/2\right)$ is a Pythagorean triple. The inscribed circle for the corresponding right triangle has radius

$$\begin{aligned} r &= \frac{k + (k^2-1)/2 - (k^2+1)/2}{2} \\ &= \frac{k-1}{2}\,. \end{aligned}$$

As $k$ goes through the odd integers 3, 5, 7, $\cdots$, the corresponding value of $r$ goes through each of the natural numbers. Note that this Pythagorean triple is also primitive.

**5.** If $(x, y, z)$ is a Pythagorean triple, show there are integers $m$ and $n$ such that $z^2 + xy = m^2 + n^2$; show that a similar statement is true for $z^2 - xy$.

$$z^2 + xy = \left(\frac{x+y+z}{2}\right)^2 + \left(\frac{x+y-z}{2}\right)^2$$
$$z^2 - xy = \left(\frac{x-y+z}{2}\right)^2 + \left(\frac{x-y-z}{2}\right)^2$$

Note that one of $x$ and $y$ is odd as is $z$. Thus, all four numerators are even, which implies that each ratio is an integer.

**6.** For any odd integer $k$ show that there is a primitive Pythagorean Triple, $(k, l, m)$, such that $m = l + 1$. Your proof should be constructive. That is, given $k$ how do you find $l$ and $m$.
Note: $(3, 4, 5)$ and $(15, 112, 113)$ are both Pythagorean triples with the desired property.

Given $k$ set $l = \frac{k^2 - 1}{2}$, and $m = \frac{k^2 + 1}{2}$. Since $k$ is odd both $l$ and $m$ are integers, and it is an easy computation to verify that $\left(k, \frac{k^2 - 1}{2}, \frac{k^2 + 1}{2}\right)$ is a Pythagorean triple whose second and third terms differ by 1.

**7.** Suppose the $\lambda$ is a positive rational number. That is, $\lambda = \frac{m}{n}$ with $m$ and $n$ positive integers. Find necessary and sufficient conditions on $m$ and $n$ such that $\sqrt{\lambda^2 + 1}$ is also rational.

A necessary and sufficient condition is that $m$ and $n$ are the smaller two integers in a Pythagorean triple. Suppose first that there is a $z \in N$ such that $m^2 + n^2 = z^2$, then we have

$$\left(\frac{m}{n}\right)^2 + 1 = \left(\frac{z}{n}\right)^2$$
$$\sqrt{\lambda^2 + 1} = \frac{z}{n}.$$

Conversely, suppose that $\sqrt{\lambda^2 + 1}$ is a rational number $\frac{a}{b}$ in lowest terms. Then

$$\frac{m^2}{n^2} + 1 = \frac{a^2}{b^2}$$
$$m^2 b^2 + n^2 b^2 = a^2 n^2$$

Since $a$ and $b$ are in lowest terms and $b^2$ divides $a^2 n^2$ we know that $b^2$ divides $n^2$, which implies that $b$ divides $n$. Thus, there is a $k$ such that $n = bk$, and we see that

$$m^2 b^2 + n^2 b^2 = a^2 b^2 k^2$$
$$m^2 + n^2 = (ak)^2.$$

That is, $m$ and $n$ are the smaller two numbers of a Pythagorean triple.

**8**. Show that if $(x, y, z)$ is a primitive Pythagorean Triple (assume that $y$ is even) then there are integers $r$ and $s$ of opposite parity such that

$$x = r^2 - s^2, y = 2rs, z = r^2 + s^2,$$

and the greatest common divisor of $r$ and $s$ is 1.

Since we have a primitive Pythagorean triple the integers $x$ and $z$ are relatively prime, which implies that the integers $\frac{z-x}{2}$ and $\frac{z+x}{2}$ are also relatively prime. Remember, we are assuming that $y$ is even, which forces $x$ and $z$ to be odd. Thus ,we have

$$\left(\frac{y}{2}\right)^2 = \frac{z+x}{2}\frac{z-x}{2}.$$

Since the integers $\frac{z-x}{2}$ and $\frac{z+x}{2}$ are relatively prime, and each divides the square of an integer, their square roots must also be integers. That is, $r = \sqrt{\frac{z+x}{2}}$ and $s = \sqrt{\frac{z-x}{2}}$ are integers, and we have

$$y = 2rs, x = r^2 - s^2, \text{ and } z = r^2 + s^2.$$

Since $x$ and $z$ are both odd $r$ and $s$ must be of opposite parity, and since $x$, $y$, and $z$ have no common divisors neither can $r$ and $s$.

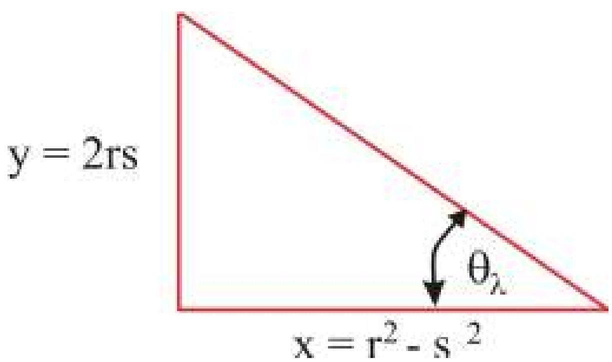
**9**. Prove that there is no isosceles right triangle with sides of integer lengths, but given any $\epsilon > 0$, there is a Pythagorean triple, such that the right triangle with side lengths given by the triple has the property that the smaller of the non-right angles is within $\epsilon$ of $\frac{\pi}{4}$.

Suppose we had an isosceles right triangle with sides of length $a$, $a$, and $b$. Then we have the equation

$$2a^2 = b^2,$$

which implies that $\sqrt{2} = b/a$ a rational number, but $\sqrt{2}$ is not rational. This contradiction tells us that an isosceles right triangle cannot exist.

To see that we can get one of the acute angles of a right triangle as close as we want to 45°, we use the result of problem 8.




To have angle $\theta_\lambda$ close to $\pi/4$, we show that we can pick $r$ and $s$ so that its tangent is close to one.

$$\tan\theta_\lambda = \frac{2rs}{r^2 - s^2} = \frac{2(r/s)}{(r/s)^2 - 1}.$$

Let $\lambda = r/s$. Then we want to pick $\lambda$ so that $\frac{2\lambda}{\lambda^2 - 1}$ is close to one. Suppose this ratio is equal to 1. Then $\lambda$ must equal $1 \pm \sqrt{2}$. Since $\lambda$ is the ratio of two positive integers, we want $\lambda$ equal to $1 + \sqrt{2}$. This can't happen for rational $\lambda$, but we can pick a sequence of rational numbers $\lambda_n$ that converge to $1 + \sqrt{2}$. That is, we can pick a sequence of positive integers $r_n$ and $s_n$ such that $\lambda_n = \frac{r_n}{s_n}$ is arbitrarily close to $1 + \sqrt{2}$, which means that $\tan\theta_{\lambda_n}$ is arbitrarily close to 1, and hence $\theta_{\lambda_n}$ is as close to $\pi/4$ as desired.