2008 Power Team Rules

1. Each power team entry must have a cover sheet (typed). The cover sheet must contain the name of the school, the team name, and the names of each team member. For example, if a school has 3 power team entries, then there should be three different team names. The following is an example of an acceptable title page:

2008 Power Team Entry

XXXX School, Team 1

John Smith

Jane Doe

etc.

2. Team participants are not allowed to consult with anyone but their team mates. Participants are not allowed to look on the web for any information regarding the power team exam, nor are they allowed to search books or other reference materials.

3. Team entries are expected to be neat and legible. If not, they face the possibility of being disqualified by the judges.

4. All hand delivered submissions are due by 9:15 am on the day of the contest! FAXed submissions will be accepted provided they are received no later than 8:30 AM. on the day of the contest. FAXed solutions should be sent to 979-862-4190, attention Mike Stecher.

2008 High School Math Contest
Power Team Exam

A triple of positive integers (x, y, z) is called a Pythagorean Triple if these integers satisfy the equation

$$x^2 + y^2 = z^2 .$$

For example, $(3, 4, 5)$ is a Pythagorean Triple. A Pythagorean Triple is said to be primitive if the integers are relatively prime. That is, they have no common integer factor except 1. The triple $(3, 4, 5)$ is also a primitive Pythagorean Triple.

1. If (x, y, z) is a primitive Pythagorean Triple show that
   a. Exactly one of x and y is odd and hence that z must also be odd,
   b. Exactly one of x and y is divisible by 3,
   c. Exactly one of x, y, and z is divisible by 5.

2. Is there a Pythagorean triple of the form $(2, y, z)$?

3. If (x, y, z) is a Pythagorean triple then the isosceles triangles with sides of length $2x, z, z$ and $2y, z, z$ have the same area.

4. Let (x, y, z) be a Pythagorean triple. Let r be the radius of the inscribed circle of the right triangle with sides x, y, and z.
   a. Show that r must be an integer with the value

$$r = \frac{xy}{x + y + z}$$

   b. Let n be any positive integer. Show that there is a right triangle whose sides have integer lengths and whose inscribed circle has radius n.

5. If (x, y, z) is a Pythagorean triple, show there are integers m and n such that $z^2 + xy = m^2 + n^2$; show that a similar statement is true for $z^2 - xy$.

6. For any odd integer k show that there is a primitive Pythagorean Triple, (k, l, m), such that $m = l + 1$. Your proof should be constructive. That is, given k how do you find l and m.
Note: $(3, 4, 5)$ and $(15, 112, 113)$ are both Pythagorean triples with the desired property.

7. Suppose the λ is a positive rational number. That is, $\lambda = \frac{m}{n}$ with m and n positive integers. Find necessary and sufficient conditions on m and n such that $\sqrt{\lambda^2 + 1}$ is also rational.

8. Show that if (x, y, z) is a primitive Pythagorean Triple (assume that y is even) then there are integers r and s of opposite parity such that

$$x = r^2 - s^2, y = 2rs, z = r^2 + s^2,$$

and the greatest common divisor of r and s is 1.

9. Prove that there is no isosceles right triangle with sides of integer length, but given any $\epsilon > 0$, there is a Pythagorean triple, such that the right triangle with side lengths given by the triple has the property that the smaller of the non-right angles is within ϵ of $\frac{\pi}{4}$.