2009 Power Team Rules

1. Each power team entry must have a cover sheet (typed). The cover sheet must contain the name of the school, the team name, and the names of each team member. For example, if a school has 3 power team entries, then there should be three different team names. The following is an example of an acceptable title page:

   2009 Power Team Entry

   XXXX School, Team 1

   Team Members

   Jane Doe

   John Smith

   etc.

2. Team participants are not allowed to consult with anyone but their team mates. Participants are not allowed to look on the web for any information regarding the power team exam, nor are they allowed to search books or other reference materials.

3. Team entries are expected to be neat and legible. If not, they face the possibility of being disqualified by the judges.

4. All hand delivered submissions are due by 9:15 am on the day of the contest! FAXed submissions will be accepted provided they are received no later than 8:30 AM. on the day of the contest. FAXed solutions should be sent to 979-862-4190, attention Mike Stecher.

2009 TAMU High School Math Contest

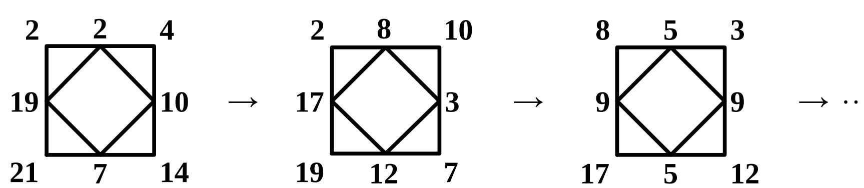
Power Team Exam

Polygon Differencing Games

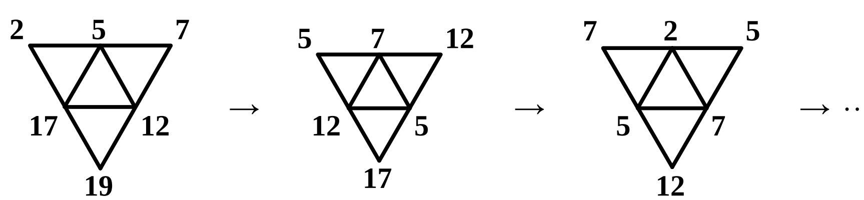
Preliminaries:

1. Draw a square and write a (small) non-negative integer at each vertex. (See the example diagram below.)

- **a.** Write the (absolute value of the) differences at the midpoints of each side.
- **b.** Connect the midpoints to produce a new square with a non-negative integer at each vertex. To get the square to line up with the original square, rotate it counterclockwise by $45°$ and enlarge slightly.
- **c.** Repeat (a) and (b) until the process "terminates". This is called a Square Differencing Game.
- **d.** Repeat the Square Differencing Game with different initial numbers.
- **e.** What do you conjecture?



2. Repeat the above process but starting with an equilateral triangle. This is called a Triangle Differencing Game. What do you conjecture?



3. Repeat the above process but starting with other regular polygons. If the polygon has N sides, this is called an N-gon Differencing Game. What do you conjecture?

Some Notation:

A configuration of an N-gon can be described as an N-tuple (an ordered list of N numbers) $\vec{a} = (a_1, a_2, \cdots, a_N)$ where $a_1, a_2, \ldots, a_N$ are the numbers on the vertices starting at a fixed vertex (say top left) and going clockwise around the N-gon. The differencing operation on an N-gon takes the configuration $\vec{a} = (a_1, a_2, \cdots, a_N)$ to the configuration

$$D\vec{a} = (|a_1 - a_2|, |a_2 - a_3|, \ldots, |a_N - a_1|).$$

We write D^p for the operation of applying D, the differencing operator, p times. We also write $G(\vec{a})$ for the game starting from $\vec{a}$. Thus $G(\vec{a})$ is the sequence of configurations

$$\vec{a}, \quad D\vec{a}, \quad D^2\vec{a}, \quad D^3\vec{a}, \quad D^4\vec{a}, \quad \cdots$$

Definitions:

A differencing game ends in a **cycle** if a configuration appears again later in the sequence of configurations. The **minimal period** of a cycle is the number of steps between two successive occurances of the same configuration. A cycle is **trivial** if its minimal period is one, that is $D\vec{a} = \vec{a}$, all the configurations are the same. You will eventually prove that every differencing game ends in a cycle. The **length** of the game is the number of steps until the game enters a cycle.Thus every N-gon Differencing Game has two interesting numbers: its length and the minimal period of the cycle it enters.

The Power Team Exam:

You are expected to prove each of the following statements or answer the following questions and prove the answers. These questions aim at finding those N's for which the N-gon Differencing Games usually end with a trivial cycle or usually end with a non-trivial cycle. In proving each statement, you may use any or all of the previous statements, whether or not you are able to prove the previous statements.

Questions to Answer and Prove:

1. Prove every N-gon Differencing Game ends in a cycle, trivial or non-trivial.

2. Let $k\vec{a} = (ka_1, ka_2, \ldots, ka_N)$ for some positive integer k. Prove $G(k\vec{a})$ has the same length and minimal period as $G(\vec{a})$.

3. Let m be the smallest of the numbers $a_1, a_2, \ldots, a_N$, and let $\vec{a} + k = (a_1 + k, a_2 + k, \ldots, a_N + k)$ for some integer $k \geq -m$. Prove $G(\vec{a} + k)$ has the same length and minimal period as $G(\vec{a})$, with the exception that if $G(\vec{a})$ has length 0 then $G(\vec{a} + k)$ has length 1.

4. Prove there is only one trivial cycle. Identify it.

5. Prove if an N-gon Differencing Game terminates in the trivial cycle, then either

- **a**. N is even or
- **b**. N is odd and the entries in the initial configuration are all equal: $(a_1, a_2, \ldots, a_N) = (a, a, \ldots, a)$.

6. If N is odd, which games must end in the trivial cycle and which must end in a non-trivial cycle? Prove it.

7. Given the initial configuration $\vec{a} = (a_1, a_2, \ldots, a_N)$ of an N-gon Differencing Game, construct the initial configuration for a rN-gon Differencing Game which has the same length and minimal period, for arbitrary positive integer r. For example, given a Pentagon Differencing Game, explain how to construct a Decagon Differencing Game and a 15-gon Differencing Game, etc, all of which have the same length and minimal period. Prove it has the same length and minimal period.

8. Prove if N is a multiple of an odd number, then there are N-gon Differencing Games which end in a non-trivial cycle. Give concrete examples when $N = 6$ and 10.

9. Prove if N is an even number, then there are N-gon Differencing Games with length greater than 1 which end in the trivial cycle. Give concrete examples when $N = 6$ and 12.

So far we have seen that if N is odd or a multiple of an odd number then the game can end in a non-trivial cycle. What about a number which is not a multiple of an odd number? Questions 11 and 13 will answer this question. First notice that if N is not a multiple of an odd number then it has no odd prime factors and must have the form $N = 2^n$.

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10. For the Square (i.e. $N = 4$) Differencing Game, for any configuration $\vec{a}$, prove the configuration $D^4\vec{a}$ has all even entries.
HINT: Prove $|p - q| \equiv (p + q) \bmod 2$?

11. Prove every Square Differencing Game terminates in the trivial cycle and find an upper bound on the length of the Square Differencing Game $G(\vec{a})$.

12. If $N = 2^n$, find a number p so that for every the N-gon Differencing Game, for any configuration $\vec{a}$, the configuration $D^p\vec{a}$ has all even entries.

13. If $N = 2^n$, prove every N-gon Differencing Game terminates in the trivial cycle and find an upper bound on the length of the N-gon Differencing Game $G(\vec{a})$.

You are NOT expected to answer the following Research Questions. Some are open questions. They are provided to give you a summary of the type of questions a mathematician would ask next.

Research Questions:

1. Can the length of a Square Differencing Game be arbitrarily large? In other words, for each positive integer L, is there a Square Differencing Game $G(\vec{a})$ of length L?

2. Suppose $N = 2^n$. Can the length of an N-gon Differencing Game be arbitrarily large? In other words, for each positive integer L, is there an N-gon Differencing Game $G(\vec{a})$ of length L?

3. Is there an upper bound on the length of a Triangle Differencing Game possibly depending on the initial configuration $\vec{a}$? Can the length of a Triangle Differencing Game be arbitrarily large? What are all possible final cycles? What are the minimal periods of the final cycles?

4. Suppose N is an odd number. Is there an upper bound on the length of an N-gon Differencing Game possibly depending on the initial configuration $\vec{a}$? Can the length of an N-gon Differencing Game be arbitrarily large? What are all possible final cycles? What are the minimal periods of the final cycles?

5. Suppose N is an even multiple of an odd number. Is there an upper bound on the length of an N-gon Differencing Game possibly depending on the initial configuration $\vec{a}$? Can the length of an N-gon Differencing Game be arbitrarily large? What are all possible final cycles? What are the minimal periods of the final cycles?