

2009 TAMU High School Math Contest

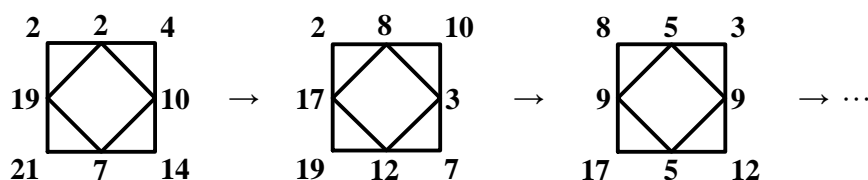
Power Team Exam

Polygon Differencing Games

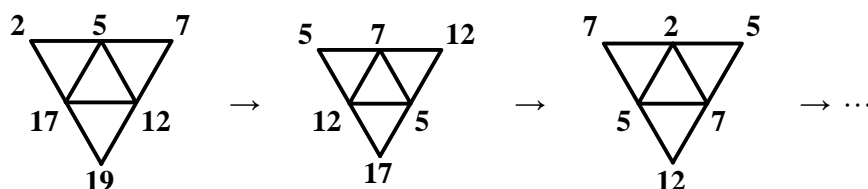
Solutions

Preliminaries:

1. Draw a square and write a (small) non-negative integer at each vertex. (See the example diagram below.)
 - a. Write the (absolute value of the) differences at the midpoints of each side.
 - b. Connect the midpoints to produce a new square with a non-negative integer at each vertex. To get the square to line up with the original square, rotate it counterclockwise by 45° and enlarge slightly.
 - c. Repeat (a) and (b) until the process "terminates". This is called a Square Differencing Game.
 - d. Repeat the Square Differencing Game with different initial numbers.
 - e. What do you conjecture?



2. Repeat the above process but starting with an equilateral triangle. This is called a Triangle Differencing Game. What do you conjecture?



3. Repeat the above process but starting with other regular polygons. If the polygon has N sides, this is called an N -gon Differencing Game. What do you conjecture?

Some Notation:

A configuration of an N -gon can be described as an N -tuple (an ordered list of N numbers) $\vec{a} = (a_1, a_2, \dots, a_N)$ where a_1, a_2, \dots, a_N are the numbers on the vertices starting at a fixed vertex (say top left) and going clockwise around the N -gon. The differencing operation on an N -gon takes the configuration $\vec{a} = (a_1, a_2, \dots, a_N)$ to the configuration

$$D\vec{a} = (|a_1 - a_2|, |a_2 - a_3|, \dots, |a_N - a_1|).$$

We write D^p for the operation of applying D , the differencing operator, p times. We also write $G(\vec{a})$ for the game starting from \vec{a} . Thus $G(\vec{a})$ is the sequence of configurations

$$\vec{a}, D\vec{a}, D^2\vec{a}, D^3\vec{a}, D^4\vec{a}, \dots$$

Definitions:

A differencing game ends in a **cycle** if a configuration appears again later in the sequence of configurations. The **minimal period** of a cycle is the number of steps between two successive occurrences of the same configuration. A cycle is **trivial** if its minimal period is one, that is $D\vec{a} = \vec{a}$, all the configurations are the same. You will eventually prove that every differencing game ends in a cycle. The **length** of the game is the number of steps until the game enters a cycle. Thus every N -gon Differencing Game has two interesting numbers: its length and the minimal period of the cycle it enters.

The Power Team Exam:

You are expected to prove each of the following statements or answer the following questions and prove the answers. These questions aim at finding those N 's for which the N -gon Differencing Games usually end with a trivial cycle or usually end with a non-trivial cycle. In proving each statement, you may use any or all of the previous statements, whether or not you are able to prove the previous statements.

Questions to Answer and Prove:

1. Prove every N -gon Differencing Game ends in a cycle, trivial or non-trivial.
 - **Lemma:** The maximal number in the configuration $D\vec{a}$ is always less than or equal to the maximal number in the configuration \vec{a} .

This follows from the fact that the difference between two non-negative integers is less than or equal to either of the numbers. Thus each entry in $D\vec{a}$ is less than or equal to an entry in \vec{a} which in turn is less than or equal to the maximal number in the configuration \vec{a} .
 - **Proof:** Let M be the maximal number in the initial configuration of an N -gon game. The differencing operation moves around within the set of configurations with numbers all less than or equal to M . Since there are $(M + 1)^N$ such configurations ($M + 1$ possible numbers, including zero, for each of N vertices) which is finite, some configuration must eventually repeat. So there is a cycle. This is an infinite version of the pigeon hole principle.
2. Let $k\vec{a} = (ka_1, ka_2, \dots, ka_N)$ for some positive integer k . Prove $G(k\vec{a})$ has the same length and minimal period as $G(\vec{a})$.

- **Proof:** First notice:

$$\begin{aligned} D(k\vec{a}) &= (|ka_1 - ka_2|, |ka_2 - ka_3|, \dots, |ka_N - ka_1|) \\ &= (k|a_1 - a_2|, k|a_2 - a_3|, \dots, k|a_N - a_1|) = kD\vec{a} \end{aligned}$$

So every configuration in the game $G(k\vec{a})$ is k times the corresponding configuration in the game $G(\vec{a})$. So if a configuration repeats in $G(\vec{a})$, then the corresponding configuration repeats in $G(k\vec{a})$. Consequently, the number of steps before the first repeating configuration (the length) and the number of steps between repeats (the period) must be the same.

3. Let m be the smallest of the numbers a_1, a_2, \dots, a_N , and let $\vec{a} + k = (a_1 + k, a_2 + k, \dots, a_N + k)$ for some integer $k \geq -m$. Prove $G(\vec{a} + k)$ has the same length and minimal period as $G(\vec{a})$, with the exception that if $G(\vec{a})$ has length 0 then $G(\vec{a} + k)$ has length 1.

● **Proof:** First notice:

$$\begin{aligned} D(\vec{a} + k) &= (|(a_1 + k) - (a_2 + k)|, |(a_2 + k) - (a_3 + k)|, \dots, |(a_N + k) - (a_1 + k)|) \\ &= (|a_1 - a_2|, |a_2 - a_3|, \dots, |a_N - a_1|) = D\vec{a} \end{aligned}$$

So every configuration in the game $G(\vec{a} + k)$ is identical to the corresponding configuration in the game $G(\vec{a})$ except for the initial configuration. So length and minimal period are unchanged unless the initial configuration \vec{a} repeats. In that case, $G(\vec{a})$ has length 0 but $G(\vec{a} + k)$ has length 1.

4. Prove there is only one trivial cycle. Identify it.

● **Answer:** The only trivial cycle has the configuration with all zeros: $\vec{0} = (0, 0, \dots, 0)$.

● **Proof:** A configuration \vec{a} starts a trivial cycle if $D\vec{a} = \vec{a}$. In other words,

$$|a_1 - a_2| = a_1, \quad |a_2 - a_3| = a_2, \quad \dots, \quad |a_N - a_1| = a_N$$

If a_i is the largest of a_1, a_2, \dots, a_N , then $a_i - a_{i+1} = a_i$, where we do not need the absolute value because a_i is the largest. So $a_{i+1} = 0$. Then $|a_{i+1} - a_{i+2}| = a_{i+1}$ implies $a_{i+2} = 0$. Etc. Hence, $\vec{a} = \vec{0} = (0, 0, \dots, 0)$.

5. Prove if an N -gon Differencing Game terminates in the trivial cycle, then either

a. N is even or

b. N is odd and the entries in the initial configuration are all equal:

$$(a_1, a_2, \dots, a_N) = (a, a, \dots, a).$$

● **Proof:** If \vec{a} is the configuration just before entering the trivial cycle, then $D\vec{a} = \vec{0}$.

So $|a_1 - a_2| = |a_2 - a_3| = \dots = |a_{N-1} - a_N| = 0$ and all the entries are equal:

$$(a_1, a_2, \dots, a_N) = (a, a, \dots, a).$$

If \vec{b} is the configuration before \vec{a} , then $D\vec{b} = \vec{a}$. So

$$|b_1 - b_2| = |b_2 - b_3| = \dots = |b_{N-1} - b_N| = |b_N - b_1| = a$$

Thus as you move through the numbers $b_1, b_2, \dots, b_N, b_1$, the numbers must increase or decrease by a at each step and eventually come back to the start. The number of steps up must equal the number of steps down. So either N is even or there is no such configuration \vec{b} , that is, $\vec{a} = (a, a, \dots, a)$ is the initial configuration.

6. If N is odd, which games must end in the trivial cycle and which must end in a non-trivial cycle? Prove it.

● **Answer:** If the entries in the initial configuration are all equal, $\vec{a} = (a, a, \dots, a)$, then the game ends in the trivial cycle. Otherwise it ends in a non-trivial cycle.

● **Proof:** If the entries in the initial configuration are all equal, then $\vec{a} = (a, a, \dots, a)$, and $D(a, a, \dots, a) = (0, 0, \dots, 0)$ which is the trivial cycle.

If the entries in the initial configuration are not all equal, but the game ends in the trivial cycle, then result 5 says N is even which is a contradiction. So the game must end in a non-trivial cycle.

7. Given the initial configuration $\vec{a} = (a_1, a_2, \dots, a_N)$ of an N -gon Differencing Game, construct the initial configuration for a rN -gon Differencing Game which has the same length and minimal period, for arbitrary positive integer r . For example, given a Pentagon Differencing Game, explain how to construct a Decagon Differencing Game and a 15-gon Differencing Game, etc, all of which have the same length and minimal period. Prove it has the same length and minimal period.

- **Answer:** The initial configuration of the rN -gon Differencing Game is $\vec{b} = (a_1, a_2, \dots, a_N, a_1, a_2, \dots, a_N, \dots, a_1, a_2, \dots, a_N,)$ where a_1, a_2, \dots, a_N is replicated r times. We will write $\vec{b} = \text{rep}_r \vec{a}$.

- **Proof:** First notice:

$$D(\vec{b}) = (|a_1 - a_2|, |a_2 - a_3|, \dots, |a_N - a_1|, |a_1 - a_2|, |a_2 - a_3|, \dots, |a_N - a_1|, \dots, |a_1 - a_2|, |a_2 - a_3|, \dots, |a_N - a_1|)$$

$$= \text{rep}_r D\vec{a}$$

So every configuration in the game $G(\vec{b})$ is the corresponding configuration in the game $G(\vec{a})$ replicated r times. So if a configuration repeats in $G(\vec{a})$, then the corresponding configuration repeats in $G(\vec{b})$. Consequently, the length and period must be the same.

8. Prove if N is a multiple of an odd number, then there are N -gon Differencing Games which end in a non-trivial cycle. Give concrete examples when $N = 6$ and 10.

- **Proof:** If N is odd, then $N = rK$ where K is odd. By result 6, there is a K -gon Differencing Game with initial configuration \vec{a} which ends in a non-trivial cycle. Then $\text{rep}_r \vec{a}$ is an N -gon Differencing Game with the same length and minimal period. So $\text{rep}_r \vec{a}$ also ends in a non-trivial cycle.

- **Example:** $N = 6$:

	(1 2 3 1 2 3)
Given the Triangle game starting with	(1 1 2 1 1 2)
$\vec{a} = (1, 2, 3)$,	(0 1 1 0 1 1)
we build the Hexagon game starting with	(1 0 1 1 0 1)
$\text{rep}_2 \vec{a} = (1, 2, 3, 1, 2, 3)$ which ends in a	(1 1 0 1 1 0)
cycle:	(0 1 1 0 1 1)

- **Example:** $N = 6$:

	(2 3 4 3 2 2)
Here is another Hexagon game which	(1 1 1 1 0 0)
ends in a cycle but is not built on a	(0 0 0 1 0 1)
Triangle game.	(0 0 1 1 1 1)
	(0 1 0 0 0 1)
	(1 1 0 0 1 1)
	(0 1 0 1 0 0)
	(1 1 1 1 0 0)

- Example:** $N = 10$:

Given the Pentagon game starting with	(1 2 3 2 1 1 2 3 2 1)
$\vec{a} = (1, 2, 3, 2, 1)$,	(1 1 1 1 0 1 1 1 1 0)
we build the Decagon game starting with	(0 0 0 1 1 0 0 0 1 1)
$\text{rep}_2 \vec{a} = (1, 2, 3, 2, 1, 1, 2, 3, 2, 1)$:	(0 0 1 0 1 0 0 1 0 1)
	(0 1 1 1 1 0 1 1 1 1)
	(1 0 0 0 1 1 0 0 0 1)
	(1 0 0 1 0 1 0 0 1 0)
	(1 0 1 1 1 1 0 1 1 1)
	(1 1 0 0 0 1 1 0 0 0)
	(0 1 0 0 1 0 1 0 0 1)
	(1 1 0 1 1 1 1 0 1 1)
	(0 1 1 0 0 0 1 1 0 0)
	(1 0 1 0 0 1 0 1 0 0)
	(1 1 1 0 1 1 1 1 0 1)
	(0 0 1 1 0 0 0 1 1 0)
	(0 1 0 1 0 0 1 0 1 0)
	(1 1 1 1 0 1 1 1 1 0)

9. Prove if N is an even number, then there are N -gon Differencing Games with length greater than 1 which end in the trivial cycle. Give concrete examples when $N = 6$ and 12.

- Proof:** Before looking at general even N , let's consider $N = 2$. Even though there are no 2-sided polygons, we can still consider 2-gon Differencing Games. If the initial configuration is $\vec{a} = (p, q)$, then $D\vec{a} = (|p - q|, |q - p|)$ and $D^2\vec{a} = (0, 0)$. So the game has length 2 as long as $p \neq q$. Now consider a general even N . Then $N = 2r$ and the initial configuration $\text{rep}_r \vec{a} = (p, q, p, q, \dots, p, q)$ where p, q is replicated r times, leads to a game which also has length 2 and ends in $\vec{0}$.

- Example:** $N = 6$:

Given the 2-gon game starting with	(1 3 1 3 1 3)
$\vec{a} = (1, 3)$,	(2 2 2 2 2 2)
we build the Hexagon game starting with	(0 0 0 0 0 0)
$\text{rep}_2 \vec{a} = (1, 3, 1, 3, 1, 3)$ which ends in the trivial cycle:	

- Example:** $N = 6$:

Here is another Hexagon game which	(1 2 3 4 3 2)
ends in a trivial cycle but is not built	(1 1 1 1 1 1)
on a 2-gon game.	(0 0 0 0 0 0)

- **Example:** $N = 12$:

Given the Square game starting with	(1 2 3 4 1 2 3 4 1 2 3 4)
$\vec{a} = (1, 2, 3, 4)$,	(1 1 1 3 1 1 1 3 1 1 1 3)
we build the Dodecagon game starting with	(0 0 2 2 0 0 2 2 0 0 2 2)
$\text{rep}_3 \vec{a} = (1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4)$:	(0 2 0 2 0 2 0 2 0 2 0 2)
	(2 2 2 2 2 2 2 2 2 2 2 2)
	(0 0 0 0 0 0 0 0 0 0 0 0)

So far we have seen that if N is odd or a multiple of an odd number then the game can end in a non-trivial cycle. What about a number which is not a multiple of an odd number? Questions 11 and 13 will answer this question. First notice that if N is not a multiple of an odd number then it has no odd prime factors and must have the form $N = 2^n$.

10. For the Square (i.e. $N = 4$) Differencing Game, for any configuration \vec{a} , prove the configuration $D^4 \vec{a}$ has all even entries.

HINT: Prove $|p - q| \equiv (p + q) \pmod{2}$?

- **Proof:** Since we are interested in knowing if a configuration has odd or even entries, it is sufficient to work modulo 2.

* **Lemma:** $|p - q| \equiv (p + q) \pmod{2}$

If p and q are both even or both odd then $|p - q|$ and $(p + q)$ are both even.

If one of p or q is even and the other is odd then $|p - q|$ and $(p + q)$ are both odd.

So $|p - q| \equiv (p + q) \pmod{2}$

Consequently, the differencing operator modulo 2 is

$$D\vec{a} = (|a_1 - a_2|, |a_2 - a_3|, |a_3 - a_4|, |a_4 - a_1|) \equiv (a_1 + a_2, a_2 + a_3, a_3 + a_4, a_4 + a_1) \pmod{2}.$$

If we write the configuration \vec{a} as a column vector, then the differencing operator modulo 2 can be written as a matrix product:

$$D\vec{a} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ a_2 + a_3 \\ a_3 + a_4 \\ a_4 + a_1 \end{pmatrix}$$

Let \mathbf{D} denote this matrix. Then D^p is represented by the matrix \mathbf{D}^p :

$$\mathbf{D}^2 = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \end{pmatrix} \quad \mathbf{D}^3 = \begin{pmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & 3 & 3 \\ 3 & 1 & 1 & 3 \\ 3 & 3 & 1 & 1 \end{pmatrix} \quad \mathbf{D}^4 = \begin{pmatrix} 2 & 4 & 6 & 4 \\ 4 & 2 & 4 & 6 \\ 6 & 4 & 2 & 4 \\ 4 & 6 & 4 & 2 \end{pmatrix}$$

It should be noted that each row is a row of Pascal's triangle, wrapped around after 4 columns and added back in.

Modulo 2 these matrices are:

$$\mathbf{D}^2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad \mathbf{D}^3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad \mathbf{D}^4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The fact that \mathbf{D}^4 is all zeros reflects the fact that row 4 of Pascal's triangle is all even except for the two 1's at the ends. The fact that \mathbf{D}^4 is all zeros says $D^4\vec{a} \equiv \vec{0} \pmod{2}$, and hence $D^4\vec{a}$ is all even.

11. Prove every Square Differencing Game terminates in the trivial cycle and find an upper bound on the length of the Square Differencing Game $G(\vec{a})$.

- **Proof:** Let M be the maximal number in the initial configuration \vec{a} . Since the differencing operation cannot increase the maximal number, the maximal number of $D^4\vec{a}$ is at most M . Since $D^4\vec{a}$ is all even, there is a configuration \vec{b} such that $D^4\vec{a} = 2\vec{b}$. So the maximal number in \vec{b} is at most $\frac{M}{2}$. Continuing in this fashion, there is a configuration \vec{c} such that $D^8\vec{a} = 2^2\vec{c}$ and the maximal number in \vec{c} is at most $\frac{M}{2^2}$. Now let k be the smallest integer so that $2^k > M$. Then there is a configuration \vec{f} such that $D^{4k}\vec{a} = 2^k\vec{f}$ and the maximal number in \vec{f} is at most $\frac{M}{2^k}$. However, the maximal number must be a non-negative integer and $\frac{M}{2^k} < 1$. So the maximal number in \vec{f} must be 0 and so $\vec{f} = \vec{0}$. But then $D^{4k}\vec{a} = 2^k\vec{f} = \vec{0}$. Consequently, \vec{a} ends in the trivial cycle in at most $4k$ steps. So the length of the game $G(\vec{a})$ is at most $4k$ where k is the smallest integer so that $2^k > M$.

12. If $N = 2^n$, find a number p so that for every the N -gon Differencing Game, for any configuration \vec{a} , the configuration $D^p\vec{a}$ has all even entries.

- **Answer:** We take $p = N = 2^n$. So we need to prove: For the 2^n -gon Differencing Game, for any configuration \vec{a} , the configuration $D^{2^n}\vec{a}$ has all even entries.

(Thus for an Octagon ($N = 8$) Differencing Game for any configuration \vec{a} , the configuration $D^8\vec{a}$ has all even entries.

And for an Hexadecagon ($N = 16$) Differencing Game for any configuration \vec{a} , the configuration $D^{16}\vec{a}$ has all even entries.)

- **Proof:** As in result 11, we will work modulo 2. Then the differencing operator modulo 2 is

$$D\vec{a} = (|a_1 - a_2|, |a_2 - a_3|, \dots, |a_N - a_1|) \equiv (a_1 + a_2, a_2 + a_3, \dots, a_N + a_1) \pmod{2}.$$

If we write the configuration \vec{a} as a column vector, then the differencing operator modulo 2 can be written as a matrix product:

$$D\vec{a} = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_N \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ a_2 + a_3 \\ a_3 + a_4 \\ \vdots \\ a_N + a_1 \end{pmatrix}$$

Let \mathbf{D} denote this matrix which has 1's on the principle diagonal and on the diagonal above that and a single 1 in the bottom left corner. Then D^p is represented by the matrix \mathbf{D}^p . The first couple powers are

$$\mathbf{D}^2 = \begin{pmatrix} 1 & 2 & 1 & \cdots & 0 \\ 0 & 1 & 2 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 1 & 0 & \cdots & 1 \end{pmatrix} \quad \mathbf{D}^3 = \begin{pmatrix} 1 & 3 & 3 & \cdots & 0 \\ 0 & 1 & 3 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 3 & 4 & 1 & \cdots & 1 \end{pmatrix} \quad \mathbf{D}^4 = \begin{pmatrix} 1 & 4 & 6 & \cdots & 0 \\ 0 & 1 & 4 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 4 & 6 & 4 & \cdots & 1 \end{pmatrix}$$

The first row in \mathbf{D}^k is the k^{th} row of Pascal's triangle. This moves to the right by one column on each line. When this runs off the right side, it wraps around to the initial columns. The reason this happens is that when you multiply \mathbf{D} times \mathbf{D}^k to get \mathbf{D}^{k+1} , the first line is the sum of the first 2 lines of \mathbf{D}^k which is the k^{th} row of Pascal's triangle added to itself shifted by one. This gives the $(k+1)^{\text{th}}$ row of Pascal's triangle. The same happens on the other rows but shifted to the right. The last row of \mathbf{D}^{k+1} is the sum of the last and first rows of \mathbf{D}^k .

The $(N-1)^{\text{th}}$ row of Pascal's triangle has N numbers which fill each row of the $N \times N$ matrix:

$$\mathbf{D}^{N-1} = \begin{pmatrix} 1 & \binom{N-1}{1} & \binom{N-1}{2} & \cdots & 1 \\ 1 & 1 & \binom{N-1}{1} & \cdots & N-1 \\ \binom{N-1}{N-2} & 1 & 1 & \cdots & \binom{N-1}{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{N-1}{1} & \binom{N-1}{2} & \binom{N-1}{3} & \cdots & 1 \end{pmatrix}$$

If you multiply this by \mathbf{D} , it again adds rows which produces the N^{th} row of Pascal's triangle except the 1's at the ends are added together.

$$\mathbf{D}^N = \begin{pmatrix} 2 & \binom{N}{1} & \binom{N}{2} & \cdots & \binom{N}{N-1} \\ \binom{N}{N-1} & 2 & \binom{N}{1} & \cdots & \binom{N}{N-2} \\ \binom{N}{N-2} & \binom{N}{N-1} & 2 & \cdots & \binom{N}{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{N}{1} & \binom{N}{2} & \binom{N}{3} & \cdots & 2 \end{pmatrix}$$

Now we want this modulo 2 with $N = 2^n$. A fact about Pascal's triangle is that if $N = 2^n$ then the N^{th} row of Pascal's triangle is all even except for the 1's at the end. Consequently, all the numbers in \mathbf{D}^N are even. So $\mathbf{D}^N = \mathbf{0} \pmod{2}$ and $\mathbf{D}^N \vec{a} = \vec{0} \pmod{2}$ Hence $D^N \vec{a}$ is all even for $N = 2^n$.

13. If $N = 2^n$, prove every N -gon Differencing Game terminates in the trivial cycle and find an upper bound on the length of the N -gon Differencing Game $G(\vec{a})$.

- Proof:** Let M be the maximal number in the initial configuration \vec{a} . Since the differencing operation cannot increase the maximal number, the maximal number of $D^N \vec{a}$ is at most M . Since $D^N \vec{a}$ is all even, there is a configuration \vec{b} such that $D^N \vec{a} = 2\vec{b}$. So the maximal number in \vec{b} is at most $\frac{M}{2}$. Continuing in this fashion, there is a configuration \vec{c} such that $D^{2N} \vec{a} = 2^2 \vec{c}$ and the maximal number in \vec{c} is at most $\frac{M}{2^2}$. Now let k be the smallest integer so that $2^k > M$. Then there is a configuration \vec{f} such that $D^{kN} \vec{a} = 2^k \vec{f}$ and the maximal number in \vec{f} is at most $\frac{M}{2^k}$. However, the maximal number must be a non-negative integer and $\frac{M}{2^k} < 1$. So the maximal number in \vec{f} must be 0 and so $\vec{f} = \vec{0}$. But then $D^{kN} \vec{a} = 2^k \vec{f} = \vec{0}$. Consequently, \vec{a} ends in the trivial cycle in at most kN steps. So the length of the game $G(\vec{a})$ is at most kN where k is the smallest integer so that $2^k > M$.

You are NOT expected to answer the following Research Questions. Some are open questions. They are provided to give you a summary of the type of questions a mathematician would ask next.

Research Questions:

- Can the length of a Square Differencing Game be arbitrarily large? In other words, for each positive integer L , is there a Square Differencing Game $G(\vec{a})$ of length L ?
- Suppose $N = 2^n$. Can the length of an N -gon Differencing Game be arbitrarily large? In other words, for each positive integer L , is there an N -gon Differencing Game $G(\vec{a})$ of length L ?
- Is there an upper bound on the length of a Triangle Differencing Game possibly depending on the initial configuration \vec{a} ? Can the length of a Triangle Differencing Game be arbitrarily large? What are all possible final cycles? What are the minimal periods of the final cycles?
- Suppose N is an odd number. Is there an upper bound on the length of an N -gon Differencing Game possibly depending on the initial configuration \vec{a} ? Can the length of an N -gon Differencing Game be arbitrarily large? What are all possible final cycles? What are the minimal periods of the final cycles?
- Suppose N is an even multiple of an odd number. Is there an upper bound on the length of an N -gon Differencing Game possibly depending on the initial configuration \vec{a} ? Can the length of an N -gon Differencing Game be arbitrarily large? What are all possible final cycles? What are the minimal periods of the final cycles?