

2010 Power Team Rules

- (1) Each power team entry must have a cover sheet (typed). The cover sheet must contain the name of the school, the team name, and the names of each team member. For example, if a school has 3 power team entries, then there should be three different team names. The following is an example of an acceptable title page:

2010 Power Team Entry

XXXX School, Team 1

Team Members

Jane Doe

John Smith

etc.

- (2) Team participants are not allowed to consult with anyone but their team mates. Participants are not allowed to look on the web for any information regarding the power team exam, nor are they allowed to search books or other reference materials.
- (3) Team entries are expected to be neat and legible. If not, they face the possibility of being disqualified by the judges.
- (4) All hand delivered submissions are due by 9:15 am on the day of the contest! FAXed submissions will be accepted provided they are received no later than 8:30 AM. on the day of the contest. FAXed solutions should be sent to 979-862-4190, attention Mike Stecher.

Power Team Exam

Texas A&M High School Math Contest
Oct. 23, 2010

AIRPORT CONFUSION NUMBERS

A typical design of a large airport is based on the concept of terminals, some of which are directly connected by hallways. A common graphical representation of the general outlay of a large airport represents the terminals by vertices and the connecting hallways by edges of a finite graph. For instance, the graph in Figure 1 represents an airport with 6 terminals some of which are connected by hallways. The terminals are represented by the vertices and each hallway connection by an edge between the corresponding vertices.

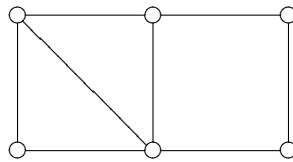


FIGURE 1. A representation of an airport with 6 terminals

For easier orientation at the airport, the terminals are usually numbered. Figure 2 represents one possible numbering of the terminals of the airport represented in Figure 1. Terminal 1 and Terminal 2 are directly connected by a hallway, while Terminal 2 and Terminal 6 are not (of course, one can walk, say, from 2 to 5 through the hallway that directly connects them, and then from 5 to 6 through their own direct connection).

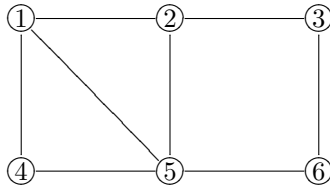


FIGURE 2. The same airport with a numbering of the terminals

There is a certain difficulty with the numbering of the terminals that is evident already in Figure 2. Namely, some terminals that are labeled by consecutive numbers are not connected by a direct hallway (such as Terminal 3 and Terminal 4), while some terminals that are connected by a direct hallway are labeled by numbers that are far apart (such as Terminal 1 and Terminal 5). The design needs

to take into account the confusion among the passengers that may arise from such inconsistencies.

Some numberings of the terminals seem to be better than others, namely those that keep the differences between the numbers labeling two adjacent terminals (terminals connected by a direct hallway) as small as possible.

Let us try to be more precise about this by introducing a measurement of the confusion caused by some particular numbering of a given layout of terminals and hallways.

For every hallway (i.e., edge in the corresponding graphical representation), the confusion number associated to that hallway is, by definition, the absolute value of the difference between the numbers labeling the two terminals (i.e., vertices in the graphical representation) connected by that hallway. In Figure 3, each edge is labeled by its confusion number. For instance, the confusion number of the edge connecting 2 and 5 is 3.

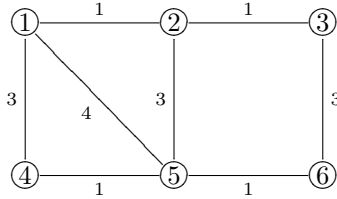


FIGURE 3. The same numbering with confusion numbers indicated on the edges

Once we determine the confusion number of every edge (hallway), we assign a confusion number to the entire numbering of the vertices (terminals) by declaring the largest confusion number of the edges to be the confusion number of the entire numbering. For instance, the confusion number of the numbering of the graph in Figure 3 is 4 (since this is the largest confusion number of one of the edges).

It is easy to see that the numbering chosen in Figure 3 is not an optimal one, at least viewed through the size of the corresponding confusion number. Indeed, we can easily provide a numbering with a smaller confusion number, namely one with confusion number 3, as indicated in Figure 4.

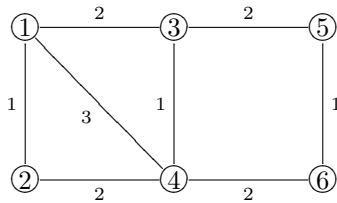


FIGURE 4. The same graph with a numbering with smaller confusion number

Among all possible numberings of a given graph, we prefer those with the smallest possible confusion number. We call that smallest possible confusion number the confusion number of the graph itself.

It is important to observe that each particular numbering of some graph has its own confusion number, but the graph itself has only one confusion number, namely the smallest confusion number of all possible numberings of its vertices.

If we consider again our example, we already have two numberings of the graph in Figure 1, one with confusion number 4 (in Figure 3) and one with confusion number 3 (in Figure 4), but the confusion number of this graph is neither 4 nor 3, since there is even better numbering of the same graph that has confusion number 2, namely the one in Figure 5.

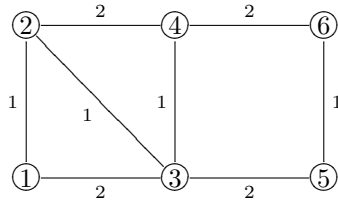
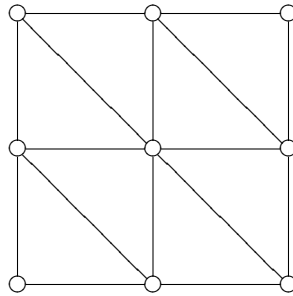


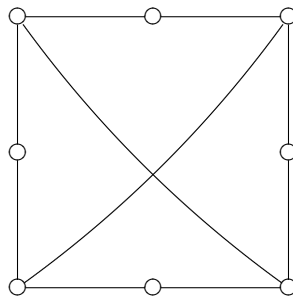
FIGURE 5. The same graph with a numbering with even smaller confusion number

Problem 1. For each of the following graphs provide a numbering with confusion number 3.

(a)



(b) (Remark: in the graph below, there is no vertex where the two curved edges meet)



In the next few problems we consider only a certain type of airport layouts. Namely, the designers of the airport in a certain city settled on a airport layout in which n terminals, where $n \geq 3$, are arranged in a circular manner in such a way that each is connected by a hallway to its two closest terminals (along the circle)

and there is an additional terminal (serving the subway) in the middle of the circle and this terminal is connected by hallways to all other terminals. In particular, n hallways meet at the central terminal, while exactly 3 hallways meet at all other (peripheral) terminals. Denote such a layout (i.e., the corresponding graph) by A_n . For instance, the graph in Figure 6 is the graph A_8 . Denote the confusion number

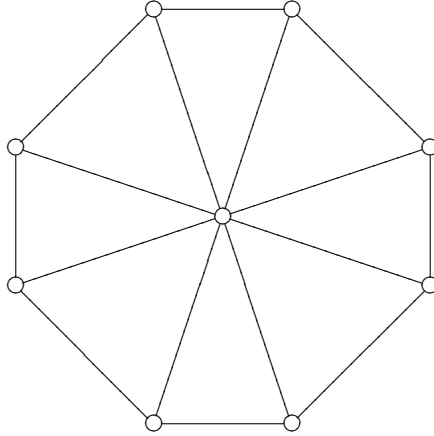


FIGURE 6. The graph A_8

of A_n by c_n .

Problem 2. Determine at least one numbering with confusion number 3 for each of the graphs A_3 , A_4 , A_5 and A_6 .

Problem 3. Determine at least one numbering with confusion number 4 for each of the graphs A_7 and A_8 .

Problem 4. Show that

$$c_3 = 3 \quad \text{and} \quad c_4 = 3.$$

(Hint: Note that the numberings provided in the solution of Problem 2 already show that $c_3 \leq 3$ and $c_4 \leq 3$. Therefore, all you need to show is that there are no numberings of A_3 and A_4 with confusion number 1 or 2).

Problem 5. Let $n \geq 3$.

(a) Show that, regardless of the numbering of the terminals (vertices) in A_n , one can get from Terminal 1 to any other terminal by using no more than 2 hallways (edges).

(b) Use the observation from part (a) to show that the inequality

$$(1) \quad 1 + 2c_n \geq n + 1$$

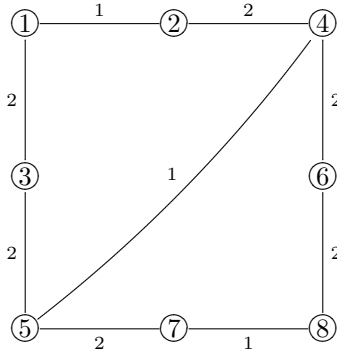
holds, where c_n is the confusion number of A_n .

Note that you may use the inequality (1) in what follows even if you did not solve Problem 5.

Problem 6. Use the previous problems to show that

$$c_5 = 3, \quad c_6 = 3, \quad c_7 = 4, \quad c_8 = 4.$$

If the confusion number of some particular layout seems to be too large (for whatever practical purpose), one may try to reduce it by modifying the underlying graph by simply deleting some of the edges. For example, it can be shown (you do not need to do it) that the graph from Problem 1(b) has confusion number 3. However, deletion of one of the “diagonal” edges leads to a graph of a smaller confusion number as can be seen from the following numbering.



Confusion number reduction through edge deletion is the theme of the remaining problems.

Problem 7. (a) Delete one edge from A_3 in such a way that the newly obtained graph has confusion number no greater than 2. (Provide a numbering of the new graph with confusion number 2.)

(b) Delete one edge from A_4 in such a way that the newly obtained graph has confusion number no greater than 2. (Provide a numbering of the new graph with confusion number 2.)

Problem 8. (a) Delete two edges from A_5 in such a way that the newly obtained graph has confusion number no greater than 2. (Provide a numbering of the new graph with confusion number 2.)

(b) Show that, no matter which single edge is deleted from A_5 , the newly obtained graph still has confusion number 3.

Problem 9. (a) Delete three edges from A_6 in such a way that the newly obtained graph has confusion number no greater than 2. (Provide a numbering of the new graph with confusion number 2.)

(b) Show that, no matter which single edge is deleted from A_6 , the newly obtained graph still has confusion number 3.

(c) Show that, no matter which two edges are deleted from A_6 , the newly obtained graph still has confusion number 3.

Problem 10. Delete two edges from the square grid graph G_5 given below and then provide a numbering of the obtained graph with confusion number 4.

