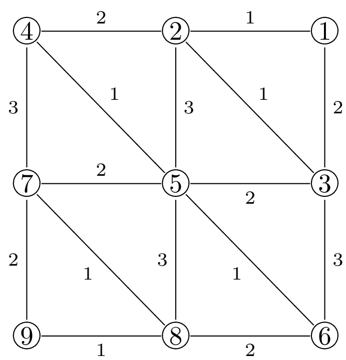
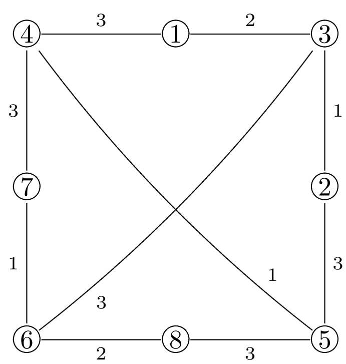
Solutions to Power Team Exam

Texas A&M High School Math Contest
Oct. 23, 2010

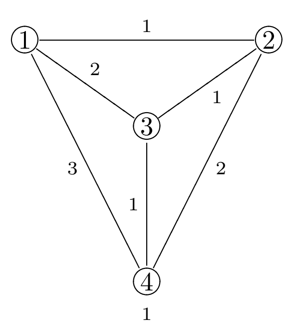
Solution 1. (a)



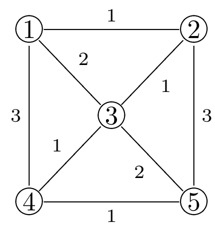
(b)



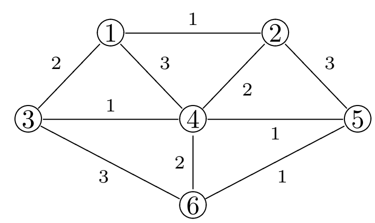
Solution 2. A numbering of A_3 with confusion number 3 is given by



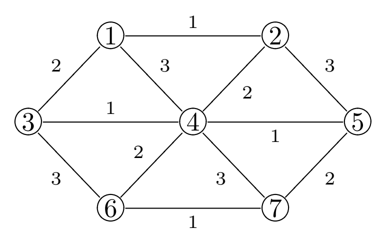
A numbering of A_4 with confusion number 3 is given by



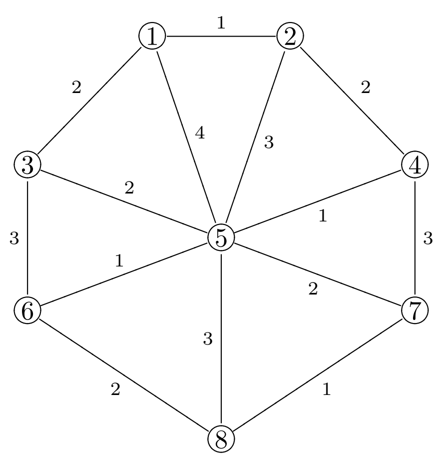
A numbering of A_5 with confusion number 3 is given by



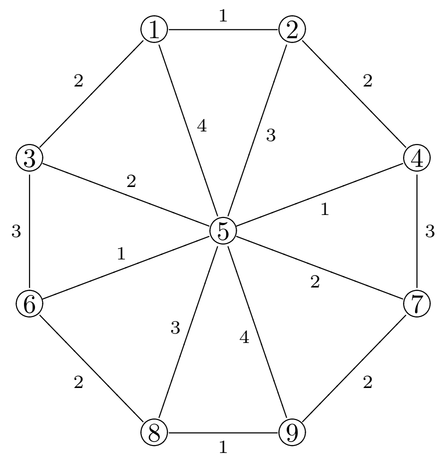
A numbering of A_6 with confusion number 3 is given by



Solution 3. A numbering of A_7 with confusion number 4 is given by



A numbering of A_8 with confusion number 4 is given by



Solution 4. Any two vertices in A_3 are connected by an edge. Since, in any numbering, one of the vertices is labeled by 1 and one is labeled by 4, the confusion number of the edge connecting them must be 3. Therefore the confusion number of any numbering of A_3 is at least 3. In fact, the confusion number of any numbering of A_3 is exactly 3 since the graph has 4 vertices and the confusion number cannot be more than $4 - 1 = 3$.

The numbering given in the solution of Problem 2 shows that the confusion number of A_4 is no greater than 3. We will show that any numbering of A_4 contains an edge with confusion number no smaller than 3.

Note that, regardless of which vertex is labeled by 1, there are at least three vertices connected to it by an edge. Therefore at least one of the vertices connected to 1 by an edge must have a label at least 4. Therefore, regardless of the numbering, there exists an edge with confusion number no smaller than 3.

Solution 5. (a) If the central vertex is labeled by 1, then we may reach any other vertex from it by using a single edge. If 1 is peripheral vertex, we may reach the central vertex by using a single edge and may reach any other peripheral vertex by using two edges, one from 1 to the central vertex and then another from the central vertex to any other peripheral vertex.

(b) Consider an optimal numbering of A_n, i.e., a labeling for which the confusion number is exactly c_n. The largest possible label on a vertex that can be reached from 1 in one step is $1 + c_n$ and the largest possible label on a vertex that can be reached from 1 in two steps is $1 + c_n + c_n = 1 + 2c_n$. Since all vertices can be reached in no more than two steps and one of them is labeled by $n + 1$ we must have

$$1 + 2c_n \geq n + 1.$$

Solution 6. Problem 5 shows that, for $n \geq 3$, $c_n \geq \frac{n}{2}$. In particular,

$$c_5 \geq 2.5, \qquad c_6 \geq 3, \qquad c_7 \geq 3.5, \qquad c_8 \geq 4.$$

Since the confusion numbers are integers, we must have

$$c_5 \geq 3, \qquad c_6 \geq 3, \qquad c_7 \geq 4, \qquad c_8 \geq 4.$$

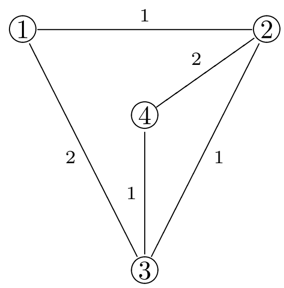
On the other hand, the numberings provided in the solutions to Problems 2 and 3 show that

$$c_5 \leq 3, \qquad c_6 \leq 3, \qquad c_7 \leq 4, \qquad c_8 \leq 4.$$

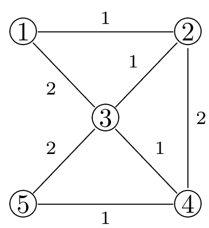
Therefore

$$c_5 = 3, \qquad c_6 = 3, \qquad c_7 = 4, \qquad c_8 = 4.$$

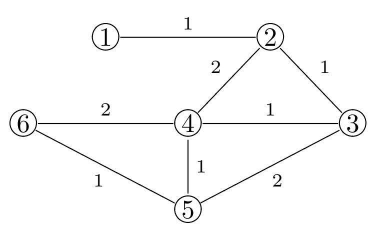
Solution 7. (a) A numbering with confusion number 2 on a graph obtained from A_3 by deletion of a single edge is given by



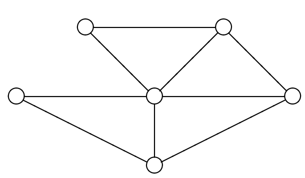
(b) A numbering with confusion number 2 on a graph obtained from A_4 by deletion of a single edge is given by

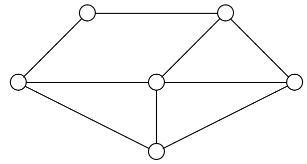


Solution 8. (a) A numbering with confusion number 2 on a graph obtained from A_5 by deletion of two edges is given by



(b) Assume that a single edge is deleted from A_5. The obtained graph then looks like one of the following two graphs



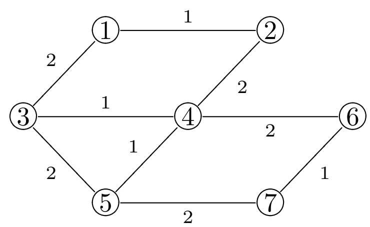


In both cases, any vertex can be reached from any other vertex in no more than 2 steps. Since there are total of 6 vertices this implies that, just as in the solution of Problem 5, that

$$1 + 2c \geq 6,$$

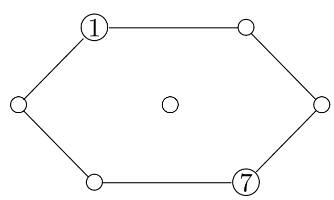
where c is the confusion number of the new graph. This inequality implies $c \geq 2.5$ and, since c is integer, $c \geq 3$.

Solution 9. (a) A numbering with confusion number 2 on a graph obtained from A_6 by deletion of three edges is given by

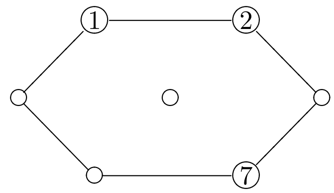


(b) The fact the the confusion number cannot be less than three when a single edge is removed from A_6, can be shown by an argument similar to that used in Problem 8(b).

(c) Assume that two edges are deleted from A_6 and the new graph has confusion number 2. Consider a numbering of the new graph with confusion number 2. It is not possible that the central vertex has 5 or 6 edges connected to it in the new graph. Indeed, for any integer x there are only 4 integers y such that $|x-y| \leq 2$. Therefore both edges that are deleted are central edges (edges connected to the central vertex) and all peripheral edges (edges connecting peripheral vertices) are still present in the new graph. Neither 1 nor 7 can be the label of the central vertex (otherwise, since the central vertex has 4 edges connected to it, the confusion number on at least one of them would be at least 4). Thus, both 1 and 7 are peripheral vertices. Moreover, 1 and 7 must be three edges apart from each other in either direction around the peripheral edges, as in the following diagram,



since the label on a vertex that can be reached in two steps from 1 in a numbering with confusion number 2 cannot be larger than 1+2+2=5. The two neighbors of 1 must be labeled by 2 and 3. Because of symmetry we may assume that the numbering looks like the one in the following diagram



The peripheral vertex that is neighbor to both 2 and 7 cannot have a label greater than 4 (since it is a neighbor of 2) not can it have a label smaller than 5 (since it is a neighbor of 7). But this means that it is impossible to choose a label on that vertex leading to a numbering with confusion number 2.

This contradiction shows that deleting two edges in A_6 is not sufficient to obtain a graph with confusion number 2.

Solution 10.