

Power Team Question - Fall 1998

INSTRUCTIONS: Work as far through these questions as you can, and ask and answer more questions about the same subject if you wish to do so. Your team will be judged on how far you get with the questions asked and how original and logical your answers and further questions are.

For any real number x , define $\lfloor x \rfloor$ to be the largest integer less than or equal to x . Thus, for example, $\lfloor \pi \rfloor = 3$.

1. Compute $\lfloor -\pi \rfloor$, $\lfloor 5 \rfloor$ and $\lfloor \sqrt{2} \rfloor$.
2. Prove: If a is an integer and x is any real number, then

$$\lfloor x + a \rfloor = \lfloor x \rfloor + a.$$

3. Let a and b be integers with $a^2b^2 + 4ab \geq 0$.

(a) Solve the following equation for u .

$$u = \frac{1}{a + u}$$

(b) Solve the following equation for u .

$$u = a + \frac{1}{b + \frac{1}{u}}$$

4. Let

$$u = \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}},$$

where the pattern continues forever. Prove that $(2 + u)u = 1$ and find u .

5. Let a be a positive integer. Derive a formula for the number u given by

$$u = a + \frac{1}{a + \frac{1}{a + \frac{1}{a + \dots}}}.$$

(where the pattern goes on forever).

6. Let a and b be integers with $a^2b^2 + 4ab \geq 0$. Find a formula for the number u given by

$$u = a + \frac{1}{b + \frac{1}{a + \frac{1}{b + \dots}}}$$

(where the pattern goes on forever).

The expressions in the previous three problems are examples of continued fractions. Continued fractions can be used to approximate numbers. Given a number x , the continued fraction expansion of x is the representation of x in the form

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

where a_0 is an integer, and a_1, a_2, a_3, \dots are positive integers. If this pattern ends at some point, we call this a “finite continued fraction,” and if it goes on forever, we call it an “infinite continued fraction.” To save space, we will often write $[a_0; a_1, a_2, a_3, \dots]$ to stand for

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

Thus,

$$[a_0; a_1, a_2, a_3, \dots] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

7. Define

$$\overline{a_1, a_2, a_3, \dots, a_k} = a_1, a_2, a_3, \dots, a_k, a_1, a_2, a_3, \dots, a_k, \dots$$

Prove that, if x is the infinite continued fraction $[a_0; \overline{a_1, a_2, a_3, \dots, a_k}]$, then x always satisfies a quadratic equation with integer coefficients.

8. Let

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

Note that $a_0 = \lfloor x \rfloor$, the greatest integer less than or equal to x .

- (a) Defining $x_1 = x - a_0$ and $x_{n+1} = 1/x_n - \lfloor 1/x_n \rfloor$ for each integer $n \geq 1$, prove that $a_n = \lfloor 1/x_n \rfloor$ for each integer $n \geq 1$.
- (b) Find a_0, a_1 , and a_2 for $x = \sqrt{5} \approx 2.23607$.
- (c) Show that after some point all the a_j are the same for the continued fraction expansion of $\sqrt{5}$.
- (d) Find the complete continued fraction expansion of $\sqrt{7}$. You may want to use the identities

$$\sqrt{7} - 1 = \frac{6}{\sqrt{7} + 1} \text{ and } \sqrt{7} - 2 = \frac{3}{\sqrt{7} + 2}.$$

9. Consider the continued fraction given by

$$[a_0; a_1, a_2, a_3, \dots, a_k, \dots].$$

Note that for each nonnegative integer n , the finite continued fraction $[a_0; a_1, a_2, a_3, \dots, a_n]$ will be a rational number. For each such n , let p_n and q_n be integers such that

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, a_3, \dots, a_n].$$

- (a) Compute p_0, p_1, q_0 , and q_1 .
- (b) Show that $p_n = a_n p_{n-1} + p_{n-2}$ and $q_n = a_n q_{n-1} + q_{n-2}$ for each integer $n \geq 2$.
- (c) Prove that $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$ for each $n \geq 1$.