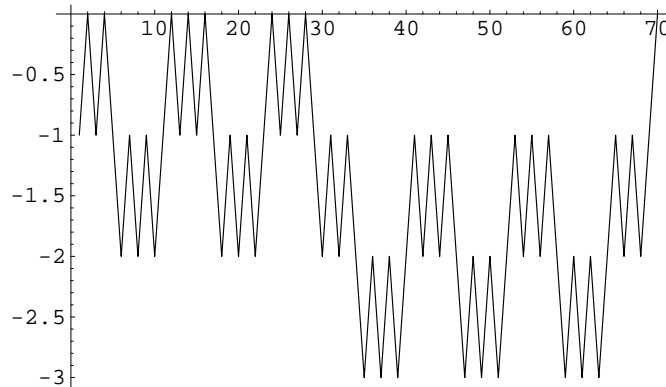


Power Contest 2001



What is going on in this figure? This is a plot of the running total net score in a game. In this game, player L scores one point each time an integer k has the property that the fractional part of $k\sqrt{2}$ is less than $1/2$, and player M scores one point each time this fractional part is more. The game starts out with player L scoring a point, which from the perspective of M is a loss. The point goes to L because the fractional part $\{\sqrt{2}\}$ of $\sqrt{2}$ is $0.41421356\dots$ and is less than $1/2$. The score returns to zero after the second move, as the fractional part of $2\sqrt{2}$ is $0.828427\dots$ which is greater than $1/2$. The game goes on and on, with M sometimes behind, and sometimes caught up to even. Does M ever take the lead? How far behind does M get during losing streaks?

To answer these questions, one must investigate along the lines laid out in the problems below. For each question, and within each question, for each part, you may take it as given that each previous question or part of a question is true, whether or not you found a proof.

1. Let $a_0 = 1, a_1 = 1$ and for $k \geq 2$ let $a_k = 2a_{k-1} + a_{k-2}$. Thus $a_2 = 3, a_3 = 7, a_4 = 17$ and so on. Similarly let $b_0 = 0, b_1 = 1$ and for $k \geq 2$ let $b_k = 2b_{k-1} + b_{k-2}$, so that $b_2 = 2, b_3 = 5, b_4 = 12$ and so on.

(a) Prove that for $k \geq 1, b_{k-1}b_{k+1} - b_k^2 = (-1)^k$. Hint: Induction.

(a') Proof: For $k = 1, 0 \cdot 2 - 1^2 = -1 = (-1)^1$, while for $k = 2, 1 \cdot 5 - 2^2 = 1 = (-1)^2$. Now suppose that the identity holds for a particular k . Then we need $b_k b_{k+2} - b_{k+1}^2 = (-1)^{k+1}$. Now $b_{k+2} = 2b_{k+1} + b_k$ so

$$\begin{aligned} b_k b_{k+2} - b_{k+1}^2 &= 2b_k b_{k+1} + b_k^2 - b_{k+1}^2 = b_{k+1}(2b_k - b_{k+1}) + b_k^2 = \\ &= b_{k+1}(2b_k - (2b_k + b_{k-1})) + b_k^2 = -(b_{k-1}b_{k+1} - b_k^2) = -(-1)^k = (-1)^{k+1} \end{aligned}$$

as required.

(b) Prove that for $k \geq 1, a_k = b_k + b_{k-1}$.

(b') Proof. The sequences a_k , and $b_k + b_{k-1}$, obey the same recursion 'next=twice previous plus the one before that.' Thus if they agree in two consecutive terms they will agree forever after. But $1 = 1 + 0$ and $3 = 2 + 1$, and for that matter, $7 = 5 + 2$.

(c) Prove that for $k \geq 1, a_k^2 - 2b_k^2 = (-1)^k$.

(c') Using part (b), $a_k = (b_k + b_{k-1})$ so

$$a_k^2 - 2b_k^2 = (b_k + b_{k-1})^2 - 2b_k^2 = 2b_k b_{k-1} + b_{k-1}^2 - b_k^2 = b_{k-1}(2b_k + b_{k-1}) - b_k^2 = b_{k-1}b_{k+1} - b_k^2 = (-1)^k$$

2. The rational numbers a_k/b_k all provide particularly good approximations to $\sqrt{2}$. You are now asked to prove this:

(a) Prove that for $k \geq 1$, $a_k/b_k < \sqrt{2}$ when k is odd, while $a_k/b_k > \sqrt{2}$ when k is even.

(a') We have $a_k^2 - 2b_k^2 = (-1)^k$ from part 1(c). Thus $(a_k/b_k)^2 - 2 = (-1)^k/b_k^2$. So if k is odd then $(a_k/b_k)^2 - 2 < 0$ so that $a_k/b_k < \sqrt{2}$, while otherwise the inequality goes the other way. Since a_k and b_k are positive, squaring or square-rooting will not change the direction of the inequalities.

(b) Prove that for $k \geq 1$,

$$\left| \frac{a_k}{b_k} - \sqrt{2} \right| < \frac{1}{2b_k^2}$$

(b') We have $(a_k/b_k)^2 - 2 = \pm b_k^{-2}$. Thus $|a_k/b_k - \sqrt{2}| \cdot |a_k/b_k + \sqrt{2}| = b_k^{-2}$. But $a_k/b_k + \sqrt{2} \geq 1 + \sqrt{2} > 2$ so $|a_k/b_k - \sqrt{2}| < 1/(2b_k^2)$ as required.

3. For certain n , the fractional part of $n\sqrt{2}$ is particularly close to $1/2$. This can complicate the determination of which side of $1/2$ such a number sits. The dangerous values of n turn out to be those n , call them n_k , so that $0 < n_k < b_k$ and $n_k a_k \equiv (b_k - 1)/2 \pmod{b_k}$. [Note: to say that $x \equiv y \pmod{z}$ is to say that $(x - y)/z$ is an integer.] Thus $n_3 = 1$ because $0 < 1 < 5$ and $1 \cdot 7 \equiv 2 \pmod{5}$. Similarly $n_5 = 6$ because $0 < 6 < b_5 = 29$ and $6 \cdot 41 \equiv 14 \pmod{29}$.

(a) Find n_7 , n_9 and n_{11} .

(a') It is not all that difficult to find n_7 by hand and without using any additional concepts; $41 \cdot 35 \equiv 14 \pmod{29}$. To persist in this approach becomes tedious though. There are two or three ways around the tedium. Least attractive, but viable, is to grind out the problem on a computer. Try 1, try 2, try 3, until you hit the magic value. Next is to use Maple. Tell it what a_k and b_k are with the defining rule, let it evaluate the various needed a_k and b_k , and then set $n_k = -(1/(2 * a_k)) \pmod{b_k}$. Modular arithmetic obeys all the usual rules of this kind of algebra, and better yet, Maple can do modular arithmetic as well as arithmetic with integers and rationals. Out pop the values of the various n_k ; they are $n_7 = 35$, $n_9 = 204$ and $n_{11} = 1189$. The same sort of thing could be done using Mathematica or Pari-GP.

(b) Guess a formula relating n_{2k+1} to n_{2k-1} and n_{2k-3} along the lines of the formulas defining a_k and b_k .

(b') The formula would seem to be $n_{2k+1} = 6n_{2k-1} - n_{2k-3}$ going from the examples.

(c) Prove that formula.

(c') First we show that $b_{k+2} = 6b_k + b_{k-2}$ from which it will follow as with part 1(b) that also $a_{k+2} = 6a_k + a_{k-2}$. Another induction. This is true by inspection for $k = 2$ and for $k = 3$, the latter by way of $29 = 6 \cdot 5 - 1 \cdot 1$. Now both sides of the induction obey the rule next = twice previous + one before that, so the identity holds for all $k \geq 2$. Next, we are led to think along these lines: n_k is the solution to $n_k a_k \equiv (-1/2) \pmod{b_k}$, so $(2n_k a_k + 1)/b_k$ should be an integer. Could it hurt, to do some arithmetic and see just which integer it is? For the case $k = 3$, we had $n_k = 1$ so $(2n_3 a_3 + 1)/b_3 = 3$. Then $(2n_5 a_5 + 1)/b_5 = 17$ and then $(2n_7 a_7 + 1)/b_7 = 99$. We've seen those numbers before. Apparently, $(2n_k a_k + 1)/b_k = a_{k-1}$ for odd k . But that is equivalent to $2n_k a_k - a_{k-1} b_k = 1$. This has just been checked for $k = 3$ and 5 and 7. What's more, we've essentially seen the n_k before. It seems that $n_3 = 1$ while $b_2 = 2$, that $n_5 = 6$ while $b_4 = 13$, that $n_7 = 35$ while $b_6 = 70$, and $n_9 = 204$ while $b_8 = 408$. Thus we decide to try to prove by induction that $n_k = (1/2)(b_{k-1})$. From this it will follow that n_k obeys the same rule as b_{k-1} and $(1/2)(b_{k-1})$. All we have to do now is to show that $b_{k-1} a_k - a_{k-1} b_k = 1$ when k is odd. But this says $b_{k-1}(b_k + b_{k-1}) - (b_{k-1} + b_{k-2})b_k = 1$ when k is odd. But $b_{k-1}^2 - b_k b_{k-2} = -(-1)^k$ whatever

the parity of k , and when k is odd, thus is 1. NOTE: The indexing was off in the first posting of this problem, with reference to even b_k instead of to odd b_k .

4. In this question, we again use the notation that $\{x\}$ denotes the fractional part of x , that is, $\{x\} = x - [x]$. Let $f(n) = -1$ if $\{n\sqrt{2}\} < 1/2$ and $f(n) = +1$ if $\{n\sqrt{2}\} > 1/2$. Prove that we would get the same values for $f(n)$, over the range $1 \leq n < b_k$, if we used the approximation a_k/b_k instead of the actual value of $\sqrt{2}$. If k is even so that b_k is even, one needs a tiebreaker rule for what to do if $na_k/b_k = 1/2$. What should that rule be?
- 4'. The point is that the error in using our approximation is less than $n/2b_k^2$ while the amount by which na_k/b_k misses the danger points 0, $1/2$, and 1 is at least $1/b_k$ if b_k is even and $na_k/b_k \neq 1/2$, and is at least $1/(2b_k)$ if b_k is odd. But this leeway is more than the possible difference between our fractional part estimate and the actual fractional part. Thus the decision will be the same using our rational approximation, and the actual value of $\sqrt{2}$. This leaves the case of b_k even and $na_k \equiv b_k/2$. But when k is even, our rational approximation to $\sqrt{2}$ is an overestimate. Thus when $n = (b_k/2)$, the fractional part of the actual $n\sqrt{2}$ is less than $1/2$. Tiebreaks in this case go to $f(n) = -1$.
5. Let $F(n) = [f(1), f(2), \dots, f(n)]$ be the sequence of ± 1 's corresponding to whether $n\sqrt{2}$ has fractional part less or more than $1/2$. You could determine $F(n)$ by evaluating successive multiples of $\sqrt{2}$. But for large n this would be tedious, even for a computer. There is, however, a pattern to $F(n)$. Despite the fact that $F(5) = [-1, +1, -1, +1, -1]$, this pattern is not pure dull alternating plus and minus. Instead, it's a pattern relating $F(b_k)$ to $F(b_{k-1})$ and $F(b_{k-2})$.

(a) Guess the pattern for the *last entry* of $F(b_k)$.

(a') The pattern is that $F(b_k)$'s last entry seems to be that the last entry is $(-1)^k$.

(b) Prove your guess.

(b') The last entry depends on which side of $1/2$ we find $b_k\sqrt{2} \bmod 1$. Now $b_k\sqrt{2} = b_k * (a_k/b_k) + b_k * (\sqrt{2} - a_k/b_k)$. The first is zero mod 1. The second has absolute value less than $1/2$ by what we proved much earlier, and it is positive if and only if k is odd because that is when $a_k/b_k < \sqrt{2}$. So mod 1, this number is between $-1/2$ and 0 (or equivalently, between $1/2$ and 1 mod 1) when k is even, and between 0 and $1/2$ when k is odd. That is, $f(b_k)$ is $+1$ when k is even and is -1 when k is odd, as guessed.

(c) Guess the pattern connecting the entire list $F(b_k)$ to the shorter lists $F(b_{k-1})$ and $F(b_{k-2})$.

(c') The pattern seems to be that this list is got by concatenating two copies of $F(b_{k-1})$ followed by a modified use of $F(b_{k-2})$. The modifications are as follows: Drop the last entry of $F(b_{k-2})$. Reverse the resulting list. Negate the entries. Tack a final entry of $(-1)^k$ on the end of your new list, and append it to the two copies of $F(b_{k-1})$. Thus, if we start out with $- + - + -$ and $- + - + - - + - + - ++$, then the next list should be got by first writing out two copies of the long list, $- + - + - - + - + - + - + - + - - + - + - ++$, then dropping the last entry from the short list to get $- + - +$, reversing it to get $+ - + -$, negating that to get $- + - +$, and tacking on a $-$ because k is odd, to get $- + - + - - + - + - + - + - + - - + - + - + - + - + -$. And indeed, this 29 entry long sequence does match the actual sequence.

The contest didn't call for a proof of this relation, but here goes. When k is even, the error in using a_k/b_k , with appropriate tiebreaking when $n = b_k/2$ or b_k or $3b_k/2$ or $2b_k$, is tolerable all the way out to $n = 2b_k$ because the difference between a half-integer and a_k/b_k is at least $1/b_k$ while the error got by using the approximation is less than $n/(2b_k^2) < 1/b_k$. After that, we can no longer trust our approximation. After all, $\sqrt{2}$ is not outright equal to a_k/b_k . Sooner or later, the two sequences must part company. When k is odd, there is a slight difficulty treating the case $n = n_k$, but the treatment of n_k given earlier allows us to deal with that. But now for the rest of the way in to $2b_k + b_{k-1} = b_{k+1}$, we can use the fact that $n\sqrt{2}$ is

closer yet to na_{k+1}/b_{k+1} . But if $n = b_{k+1} - m$, then $na_{k+1} - b_{k+1} \equiv -ma_{k+1}/b_{k+1} \pmod{1}$. This accounts for the negation and for the reversal, because the f we get, using either the exact value, or a_{k+1}/b_{k+1} , or a_{k-1}/b_{k-1} , will be the same in this short range for m . The appended final value was determined earlier.

6. Let $g(n) = f(1) + f(2) + \dots + f(n)$ and let $G(n) = [g(1), g(2), \dots, g(n)]$. On the basis of the apparent pattern for $f(n)$, what would you expect the rule to be for the smallest and largest values of $g(n)$ in the list $G(b_k)$? Explain, and give such evidence and grounds for your conjecture as you can. A proof is far from easy, but see what sort of evidence you can muster for your analysis. Check your guess against direct evaluations of $G(b_k)$ for various modest sized k to see if you're on the right track. The graph at the top of this problem set is a plot of $g(n)$ for $1 \leq n \leq 70$.

6'. The list $G(b_k)$ attains a minimum value of $-[k/2]$, ending with zero if k is even and with -1 if k is odd. This is verifiable for small k by making tables and looking at them. Now suppose we have a k just off the edge of what we've done in this fashion. $G(b_k)$ is going to be constructed using two copies of $F(b_{k-1})$ followed by that modified reversed copy of $F(b_{k-2})$. If k is even then $k-1$ is odd. The first of the three segments will reach $-(k-2)/2$ and end with -1 . The second segment will start already at -1 , and will repeat the pattern of the first to hit a minimum of $-(k-2)/2 - 1 = -(k)/2$ and end at -2 . The final segment will involve a modified copy of $F(b_{k-2})$. This will have had a $+1$ at its end, which got deleted by our rule. Thus the total changes over the course of this list will have been a net of -1 when read backwards, which is a net of $+1$ when read forwards. The final $+1$ puts us back to zero, so that $G(b_k)$ ends with a zero when k is even. When k is odd, a similar bit of logic shows that if F has the kind of pattern claimed (and it does) then G has the claimed pattern of minimal and final values.

Final remark: There is a theory of *discrepancy* in which the difference between the actual distribution of a sequence, and perfectly uniform distribution, is studied. Any finite sequence cannot be distributed totally evenly on a line, for there are only finitely many dots to put down. The discrepancy is the maximum of the absolute value of the difference between the length of an interval, and the proportion of dots falling in that interval. What we've seen here is that, if you just look at a single interval $(0, 1/2)$, then the discrepancy of our sequence of multiples of $\sqrt{2}$, spun out to depth b_k , is on the order of $\log b_k$. It so happens that for more advanced and general reasons, the discrepancy of this sequence, taking into account intervals of any length rather than just $1/2$, is also always on the order of the log of the number of terms considered so far. In this, the sequence does as well as a discrete sequence can do, because of another theorem...any discrete infinite sequence can be truncated at various finite quitting points Q_k , and using some interval $(0, x_k)$, will be greater than $C \log k$. Here, C is some fixed positive constant. I would like to say $C = 1/2$ works but I don't have the book handy to check this.

There is a rich body of knowledge about approximation of real numbers and in particular about the square roots of integers, by rational numbers. See C. D. Olds, Continued Fractions, for more about this topic.