Problem 1. Let \( m \geq n \geq k \geq 2 \) be natural numbers. A set \( M \) of \( m \) points with the property \((m, n, k)\) is given in the plane. Show that the minimal number of disks of radius 1 (diameter 2) that is always sufficient to cover all the points in \( M \) is \( n - k + 1 \). What can you say if \( k = 1 \)?

We first point out that there are at most \( n - k + 1 \) points such that the distance from a point to its nearest neighbor is more than 1. Suppose there are \( n - k + 2 \) such points. Add \( k - 2 \) points to the set. Then, since \( M \) has property \((m, n, k)\), \( k \) of these points can be covered by a disk of diameter 1. Since at least 2 of these points must come from the original set of \( n - k + 2 \) points, this contradicts the fact that the distance between these two points must be greater than 1. Thus, we cannot have \( n - k + 2 \) points whose distance from one point to any other is greater than 1.

To see that \( n - k + 1 \) disks of radius 1 is sufficient to cover any set, \( M \), with property \((m, n, k)\), pick any point \( A_1 \) of \( M \), cover it with a disk, \( D_1 \), of radius 1. If this disk covers \( M \), we are done. If not, pick a point \( A_2 \) not in \( D_1 \) cover it with a disk, \( D_2 \) of radius 1. Note that the distance between \( A_1 \) and \( A_2 \) is greater than 1. Continue in this fashion. After \( n - k + 1 \) disks are chosen, the set \( M \) must be covered, for otherwise we can find a point in \( M \), which is further than 1 unit from the first \( n - k + 1 \) points found, which are all at a distance greater than 1 from each other.

Lastly we need to exhibit a set with property \((m, n, k)\) which cannot be covered with fewer than \( n - k + 1 \) disks of radius 1. Put \( m - (n - k) \) points inside a disk of diameter 1. Then place \( n - k \) points so that the distance of each of them from the disk and from each other is greater than 2. Any collection of \( n \) points must have at least \( k \) of them inside the disk. Thus, our set has property \((m, n, k)\), and it needs \( n - k + 1 \) disks of diameter 2 to cover the entire set.

If \( k = 1 \), then the minimal number of disks if \( m \). An example of a set with property \((m, n, 1)\), that needs \( m \) disks of diameter 2 to cover it is given by \( M = \{(10i, 0)\}_{i=0}^{m-1} \).
Problem 2. Calculate the following numbers (be sure to carefully justify your answer in each case).

(2a) \( #(4, 2, 1) = 4 \)
Take any set of 4 points for which no point is closer than 2 to its nearest neighbor. In fact it is clear that no matter what the values of \( m \) and \( n \) are, we have \( #(m, n, 1) = m \)

(2b) \( #(4, 2, 2) = 2 \)
Take an equilateral triangle with edge length equal to 1, and put a forth point inside the triangle. This set has property \((4, 2, 2)\) and needs 2 disks to cover it. To see that 2 disks suffice for a set with this property. Pick any 2 points from the set. They can be covered by a disk of diameter 1. There are at most 2 points not in this disk, and they can be covered by a second disk.

(2c) \( #(4, 3, 1) = 4 \). See the answer to part (2a).

(2d) \( #(4, 3, 2) = 3 \).
To see that 3 disks are sufficient to cover a set with property \((4, 3, 2)\), note that 2 points can be covered by a single disk, which leaves at most two points not covered. Since each of these points can be covered by a single disk, we will have used at most 3 disks to cover the set.
To see that 3 disks are needed, let \( M \) be the set consisting of the vertices of an equilateral triangle with edge length 1. Put a fourth point more than 1 unit from any of the vertices. See the figure below.

This set has property \((4, 3, 2)\), for no matter what subset of 3 points is picked at least two will be triangular vertices, and any two vertices are distance 1 from each other. Since we need 2 disks to cover the triangle, and the fourth point cannot be in either of these disks, 3 disks are needed to cover the set.
Problem 3. Calculate the following numbers (be sure to carefully justify your answer in each case).

(3a) \( \#(5, 4, 2) = 4. \)

To see that 4 disks are sufficient we note that when we cover 2 points with one disk, there will be at most 3 points uncovered, and each of these can be covered with a single disk. To see that 4 disks may be required let \( M \) be a set with 5 points such that three points are the vertices of an equilateral triangle, with edge length 1. The last 2 points are placed far away from the triangle and each other. This set has property \( (5, 4, 2) \), since any subset with 4 points must contain at least 2 of the triangular vertices, which can be covered with a single disk, and it will take 4 disks to cover the set \( M \).

\[
\begin{array}{cccc}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\end{array}
\]

(3b) \( \#(5, 4, 3) = 2. \)

Problem 1. tells us that at least two disks are required, for if there is a set that requires 2 disks of radius 1 to cover it, then at least 2 disks of radius 1/2 are also needed.

To see that two disks of diameter 1 are sufficient let \( M = \{A, B, C, X, Y\} \) be a set with property \( (5, 4, 3) \) such that the points \( A, B, \) and \( C, \) can be covered by one disk, and the set \( M \) cannot be covered with 2 disks. Note this means that the points \( X \) and \( Y \) cannot be covered with a single disk.

Consider the following subsets of \( M \): \( \{A, B, X, Y\} \), \( \{A, C, X, Y\} \), and \( \{B, C, X, Y\} \). Each of these has a subset of three points, which can be covered by a single disk, and each of these subsets must contain only one of the points \( X \) or \( Y \), since by supposition these two points cannot be covered by a single disk. Denote these subsets by \( T_A, T_B, T_C \), where \( T_A = \{B, C, X\} \) or \( \{B, C, Y\} \), etc. If the points \( X \) and \( Y \) both occur in the sets \( T_A, T_B, T_C \), say \( X \in T_A \) and \( Y \in T_B \) then we have

\[
T_A \cup T_B = \{B, C, X\} \cup \{A, C, Y\} = \{A, B, C, X, Y\} = M,
\]

and \( M \) can be covered by two disks. Thus, only one of the points \( X \) or \( Y \) can be in the sets \( T_A, T_B, T_C \). Suppose it is \( X \).
Then the four sets
\{A, B, X\}, \{A, C, X\}, \{B, C, X\}, \{A, B, C\}
can all be covered by a single disk. Thus, the set \{A, B, C, X\} has prop-
erty (4, 3, 3), and by Problem 6. can be covered by a single disk, which
means that the original set \(M\) can be covered by a single disk. Thus,
the supposition that there is such a set \(M\) leads to a contradiction.
Hence we see that \(#(5, 4, 3) = 2\).

(3c) \(#(5, 4, 4) = 1\). This follows from the fact that a set with property
(5, 4, 4) also has property (5, 3, 3), and for such sets we know that the
minimal number of disks needed is 1.

**Problem 4.** Calculate the following numbers (be sure to carefully
justify your answer in each case).

(4a) \(#(6, 5, 2) = 5\).
Since 2 points can be covered with one disk leaving at most 4 points
uncovered, we see that 5 disks will suffice to cover the 6 points. A set
with property (6, 5, 2) that needs 5 disks to cover it is given by placing
3 points at the vertices of an equilateral triangle with edge length 1 and
the other three points such that they all have distance greater than 1
from each other as well as the first three points. Any subset of 5 points
must contain 2 of the triangular vertices, and they can be covered by
a disk of diameter 1.

\[
\begin{array}{ccc}
\cdot & & \\
\cdot & \cdot & \cdot \\
\end{array}
\]
(4b) #(6, 5, 3) = 3

From Problem 1 #(6, 5, 3) \geq 5 - 3 + 1 = 3. It remains to be shown that 3 disks are always sufficient.

At least 3 of the 6 points can be covered by a disk of diameter 1. Let $A$, $B$ and $C$ be three such points and denote the other 3 points by $X$, $Y$ and $Z$.

If any 2 of the points $X$, $Y$ and $Z$ are within distance 1 from each other than these two points can be covered by a disk of diameter 1 and 3 disks are sufficient to cover all 6 points $(3 + 2 + 1 = 6)$.

Assume that the mutual distances between $X$, $Y$ and $Z$ are greater than 1. Thus no disk of diameter 1 contains 2 of these three points. Consider the three sets of points $S_A = \{B, C, X, Y, Z\}$, $S_B = \{A, C, X, Y, Z\}$ and $S_C = \{A, B, X, Y, Z\}$. By the property $(6, 5, 3)$ each contains a subset of three points, denoted $T_A$, $T_B$ and $T_C$ respectively, that can be covered by a disk of diameter 1. The subsets $T_A$, $T_B$ and $T_C$ always involve exactly two of the points $A$, $B$ and $C$. If the union of any two of the sets $T_A$, $T_B$ and $T_C$ contains 5 points we again see that 3 disks of diameter 1 suffice $(5 + 1 = 6)$. Otherwise all three sets $T_A$, $T_B$ and $T_C$ contain the same point among $X$, $Y$ and $Z$. Assume, without loss of generality, that $X$ is contained in $T_A$, $T_B$ and $T_C$. Then the points $A$, $B$, $C$ and $X$ satisfy the property $(4, 3, 3)$ and, by Problem 6, can be covered by a single disk of diameter 1. Since the other 2 points can be covered by 2 disks, 3 disks suffice in each case.
(4c) $\#(6, 5, 4) = 2$

Let $M$ be a set with property $(6, 5, 4)$. We know that 4 of the 6 points can be covered by a single disk of diameter 1. Label these points A, B, C, D. Label the other 2 points X and Y. We can assume that the distance between X and Y is greater than 1 for otherwise they can be covered by a disk of diameter 1, which means we can cover $M$ with 2 disks. Consider the 4 sets: $S_A = \{B, C, D, X, Y\}, S_B = \{A, C, D, X, Y\}, S_C = \{A, B, D, X, Y\}$, and $S_D = \{A, B, C, X, Y\}$. Since each of these sets has 5 points, and $M$ has property $(6, 5, 4)$, each of these sets has a subset of 4 points that can be covered by a single disk. Moreover each of these subsets must contain exactly one of X and Y. If the union of two of these subsets contains both X and Y, then that union is all of $M$, and we have covered $M$ with 2 disks. So suppose now that only the point X is in these 4 subsets. This means that the sets

$\{B, C, D, X\}, \{A, C, D, X\}, \{A, B, D, X\}$, and $\{A, B, C, X\}$

can each be covered by a single disk. That is, the set $\{A, B, C, D, X\}$ has property $(5, 4, 4)$. Such a set also has property $(5, 3, 3)$, and thus, can be covered by a single disk. Which means that the original set $M$ can be covered by 2 disks.

To see that 2 disks are necessary, use Problem 1.
Problem 5. Four disks of diameter 1 are placed in the plane. If every proper subcollection of the four disks have a common point, then all four disks have a common point.

Let \( d_A, d_B, d_C \) and \( d_D \) be four disks of diameter 1 in the plane such that any three of them have a common point. Let \( A \) be a point in \( d_B \cap d_C \cap d_D \), \( B \) a point in \( d_A \cap d_C \cap d_D \), \( C \) a point in \( d_A \cap d_B \cap d_D \) and \( D \) a point in \( d_A \cap d_B \cap d_C \).

There are 4 possibilities for the arrangement of the four points: all 4 lie on a straight line, only 3 lie on a straight line, no 3 points lie on a single straight line. This last case has two parts, three points form a triangle with the fourth point outside the triangle, or the fourth point inside the triangle.

4 points on a line Assume that the points lie on the line in increasing alpha order.

\[
\begin{array}{cccc}
\ast & \ast & \ast & \ast \\
A & B & C & D
\end{array}
\]

The only disk of the four, which may not contain the point \( C \) is \( d_C \). However, that disk contains the points \( A \) and \( D \), and must also contain the point \( C \). Hence, the point \( C \) is in all 4 disks.

3 points on a line

\[
\begin{array}{ccc}
\ast & \ast & \ast \\
D & \\
A & B & C
\end{array}
\]

The only disk which may possibly not contain the point \( B \) is \( d_B \), but this disk contains the points \( A \) and \( C \), and hence must contain the point \( B \). Thus, all 4 disks contain the point \( B \).
Triangle. Assume one of the four points $A$, $B$ $C$ and $D$ is inside the triangle (including the degenerate cases) determined by the other three points. Without loss of generality assume that $D$ is inside the triangle $ABC$. Since $A$, $B$ and $C$ all belong to $d_D$ the whole triangle $ABC$, along with the interior point $D$ is contained in $d_D$. Thus, the point $D$ is contained in all 4 disks.

Convex quadrilateral. The four points $A$, $B$, $C$ and $D$ form a convex quadrilateral $Q$. Without loss of generality assume that $AC$ and $BD$ are the diagonals of $Q$. Since both $A$ and $C$ belong to $d_B$ and $d_D$, the whole diagonal $AC$ belongs to $d_B \cap d_D$. Similarly, the whole diagonal $BD$ belongs to $d_A \cap d_C$. But then the intersection of the two diagonals, $q$, belongs to all four disks $d_A$, $d_B$, $d_C$ and $d_D$. 
Problem 6. Four points are placed in the plane. If every proper subset of these four points can be covered by a disk of diameter 1, then all four points can be covered by such a disk.

Label the four points $A, B, C, D$. Let $d_X$ denote the disk of radius $1/2$ centered at the point $X$. Since every proper subset of these four points can be covered by a single disk of diameter one, there are points $A', B', C', D'$ such that

$$\{A, B, C\} \subset d_{A'}, \quad \{A, B, D\} \subset d_{B'}, \quad \{A, C, D\} \subset d_{A'}, \quad \{B, C, D\} \subset d_{A'}$$

Note that if $X \in d_Y$ then $Y \in d_X$. This is nothing more then saying if $X$ is within $1/2$ of $Y$, then $Y$ is within $1/2$ of $X$. Thus, we see that we have the following:

$$\{A', B', C'\} \subset d_D, \quad \{A', B', D'\} \subset d_C,$$

$$\{A', C', D'\} \subset d_B, \quad \{B', C', D'\} \subset d_A$$

These four disks, $d_A, d_B, d_C$, and $d_D$ have the property that the intersection of any three of them has a common point. For example $A' \in d_B \cap d_C \cap d_D$. Thus, by problem 5, the intersection of all 4 of these disks contains at least one point. Call this point $P$. That is, $P \in d_A \cap d_B \cap d_C \cap d_D$. This then implies that $\{A, B, C, D\} \subset d_P$. 