POWER TEAM 2013 SOLUTIONS

Texas A&M University High School Math Contest

November 16, 2013

In working the problems below you may find the following lemma, which is due to Gauss, helpful.

Lemma 1. *Let $P(x) = Q(x)R(x)$ be a monic polynomial with integer coefficients. If Q and R are both polynomials with Q a monic polynomial with integer coefficients, then R is also a monic polynomial with integer coefficients. Be sure to mention when you use this fact.*

The word monic means that the coefficient of the highest power of x is 1. The word zero or root when applied to a polynomial means a number a such that $P(a) = 0$.

Problem 1. Let $P(x)$ be a polynomial with integer coefficients. Show that if a is an integer, then $P(x) - P(a) = (x - a)Q(x)$, where $Q(x)$ is a polynomial with integer coefficients.

Solution. This follows from the fact that for every positive integer k we have:

$$x^k - a^k = (x - a)\left(x^{k-1} + x^{k-2}a + \cdots + xa^{k-2} + a^{k-1}\right).$$

□

Problem 2. Let

$$P(x) = x^5 + 10x^4 + 50x^3 + a_2x^2 + a_1x + a_0,$$

where a_0, a_1 and a_2 are some real numbers. Show that the polynomial $P(x)$ cannot have 5 real zeros.

Solution. Assume that $P(x)$ has 5 real zeros, x_1, x_2, x_3, x_4, x_5. Then

$$P(x) = (x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5)$$

and, by comparing the coefficients of x^4 and x^3 on the left and on the right in the last equality, we have

$$-x_1 - x_2 - x_3 - x_4 - x_5 = 10$$

and

$$x_1x_2 + x_1x_3 + x_1x_4 + x_1x_5 + x_2x_3 + x_2x_4 + x_2x_5 + x_3x_4 + x_3x_5 + x_4x_5 = 50.$$

Therefore

$$100 = (-x_1 - x_2 - x_3 - x_4 - x_5)^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + 2 \cdot 50,$$

which implies that $x_1 = x_2 = x_3 = x_4 = x_5 = 0$, but this is impossible, as then we would have $x_1 + x_2 + x_3 + x_4 + x_5 = 0$, which is not the case. □

Problem 3. Let $P(x)$ be a polynomial of degree 2013 with integer coefficients, such that $P(1) = P(2) = 17$. Show that for no integer a we may have $P(a) = 34$.

Proof. Consider the polynomial $Q(x) = P(x) - 17$, which has integer coefficients. We have $Q(1) = Q(2) = 0$, which means

$$Q(x) = (x-1)(x-2)R(x),$$

for some polynomial $R(x)$ of degree 2011 with integer coefficients. If $P(a) = 34$ then $Q(a) = P(a) - 17 = 34 - 17 = 17$ and we have

$$17 = (a-1)(a-2)R(a).$$

However, the last equality is impossible since the only divisors of 17 are -17,-1,1, and 17, while the number $(a-1)(a-2)R(a)$ has two consecutive integers, $a-1$ and $a-2$, as divisors. □

Problem 4. Show that there is no polynomial with integer coefficients such that there are three distinct integers a, b, and c such that

$$P(a) = b, \qquad P(b) = c, \qquad P(c) = a.$$

Solution. Assume that such a polynomial exists. Then $a-b$ divides $P(a) - P(b) = b - c$, $b - c$ divides $P(b) - P(c) = c - a$, and $c - a$ divides $P(c) - P(a) = a - b$. This implies that

$$|a-b| = |b-c| = |c-a|,$$

which is impossible for three distinct integers. □

Problem 5. Let $n = 3^{2013}$ and $m = 2^{2013}$. Consider the coefficient in front of x^m in the polynomials

$$P(x) = (1 - x^2 + x^3)^n \qquad \text{and} \qquad Q(x) = (1 + x^2 - x^3)^n.$$

Which one is larger?

Solution. Since m is even the coefficient in front of x^m in $Q(x)$ is the same as the coefficient in front of x^m in $Q(-x) = (1 + x^2 + x^3)^n$. As we multiply the terms on the right hand of the last equality and collect "like terms" they always add up, while if we multiply the terms on the right hand side of $P(x) = (1 - x^2 + x^3)^n$ some of the "like terms" show up with negative coefficients. In particular, some of the x^m-terms show up with negative coefficients provided the equation

$$2i + 3j = m$$

has a solution in non-negative integers such that i is odd. One such solution is $j = 2$ and $i = 2^{2012} - 3$. Therefore the coefficient in front of x^m in the polynomial $Q(x)$ is larger than the corresponding coefficient in $P(x)$. □

Problem 6. (a) Let

$$P(x) = (x - a_1)(x - a_2) \cdots (x - a_{2012}) - 1$$

where $a_1, a_2, \ldots, a_{2012}$ are distinct integers. Show that $P(x)$ cannot be written as a product

$$P(x) = Q(x)R(x)$$

of two nonconstant polynomials $Q(x)$ and $R(x)$ with integer coefficients.

(b) Let

$$P(x) = (x - a_1)(x - a_2) \cdots (x - a_{2013}) + 1$$

where $a_1, a_2, \ldots, a_{2013}$ are distinct integers. Show that $P(x)$ cannot be written as a product

$$P(x) = Q(x)R(x)$$

of two nonconstant polynomials $Q(x)$ and $R(x)$ with integer coefficients.

(c) Let $P(x)$ be a polynomial of degree 2013 such that there are distinct integers $a_1, a_2, \ldots, a_{2013}$ for which the value of the polynomial is 1 or -1 (1 for some of them and -1 for some of them). Show that $P(x)$ cannot be written as a product

$$P(x) = Q(x)R(x)$$

of two nonconstant polynomials $Q(x)$ and $R(x)$ with integer coefficients.

Solution. (a) Assume that it can. Then, for $i = 1, 2, \ldots, 2012$,

$$-1 = Q(a_i)R(a_i),$$

which implies, for $i = 1, 2, \ldots, 2012$,

$$Q(a_i) + R(a_i) = 0,$$

since the only way to write -1 as a product of two integers is by using 1 and -1. However, this means that the polynomial

$$Q(x) + R(x)$$

has at least 2012 distinct zeros. On the other hand this polynomial has degree smaller than 2012 (since neither $Q(x)$ nor $R(x)$ is constant and their product is of degree 2012). Therefore $Q(x) + R(x)$ is the zero constant polynomial, i.e., $R(x) = -Q(x)$. Since $P(x) = Q(x)R(x)$, we have

$$(x - a_1)(x - a_2) \cdots (x - a_{2012}) - 1 = -(Q(x))^2.$$

The last equality is impossible since the value of the polynomial on the right is always nonpositive, while the polynomial on the left has positive values for sufficiently large values of x.

(b) Assume that it can. Then, for $i = 1, 2, \ldots, 2013$,

$$1 = Q(a_i)R(a_i),$$

which implies, for $i = 1, 2, \ldots, 2013$,

$$Q(a_i) - R(a_i) = 0,$$

since the only ways to write 1 as a product of two integers are by using 1 and 1 or -1 and -1. However, this means that the polynomial

$$Q(x) - R(x)$$

has at least 2013 distinct roots. On the other hand this polynomial has degree smaller than 2013 (since neither $Q(x)$ nor $R(x)$ is constant and their product is of degree 2013). Therefore $Q(x) - R(x)$ is the zero constant polynomial, i.e., $R(x) = Q(x)$. Since $P(x) = Q(x)R(x)$, we have

$$(x - a_1)(x - a_2) \cdots (x - a_{2013}) + 1 = (Q(x))^2.$$

The last equality is impossible since the polynomial on the left is of odd degree and the polynomial on the right is of even degree.

(c) Assume that it can and that the degree of $Q(x)$ is no greater than the degree of $R(x)$. Since the degree of $P(X)$ is 2013, the degree of $Q(x)$ is no greater than 1006. The fact that, $Q(a_i)R(a_i) = P(a_i)$ is equal to 1 or -1, for $i = 1, 2, \ldots, 2013$, implies that $Q(a_i)$ is equal to 1 or -1, for $i = 1, 2, \ldots, 2013$. But this means that $Q(x)$ achieves one of these two values, 1 or -1, at no fewer than 1007 different points. Since $Q(x)$ has degree no greater than 1006, this is only possible if $Q(x)$ is a constant polynomial. □

Problem 7. Find all polynomials $P(x)$ of degree n, $n \geq 2$, with integer coefficients, such that $P(0) = 0$ and there are n distinct integers $a_1, a_2, \ldots, a_n$ with $P(a_i) = n$, for $i = 1, 2, \ldots, n$.

Solution. Consider the polynomial $Q(x) = P(x) - n$ of degree n. Since $Q(a_i) = P(a_i) - n = 0$, for $i = 1, 2, \ldots, n$, we have

$$Q(x) = A(x - a_1)(x - a_2) \ldots (x - a_n),$$

for some constant A. For $x = 0$, we obtain

$$-n = A(-1)^n a_1 a_2 \ldots a_n. \tag{1}$$

None of the distinct integers $a_1, \ldots, a_n$ can be 0, and at most 2 of them can have absolute value 1. Therefore

$$n = |A(-1)^n a_1 a_2 \ldots a_n| \geq 2^{n-2}.$$

The inequality $n \geq 2^{n-2}$ is not valid for $n \geq 5$. Indeed, using induction, for $n = 5$, $5 < 2^{5-2} = 8$, and if we assume $k < 2^{k-2}$, for some $k \geq 5$, then

$$k + 1 < k + k = 2k < 2 \cdot 2^{k-2} = 2^{(k+1)-2}.$$

It remains to check the cases $n = 2, 3, 4$. Without loss of generality, we may assume that $a_1 < a_2 < \cdots < a_n$.

For $n = 2$, equality (1) becomes

$$-2 = Aa_1a_2,$$

which has 5 solutions

$$\begin{array}{lll} A = 2, & a_1 = -1, & a_2 = 1, \\ A = -1, & a_1 = -2, & a_2 = -1, \\ A = -1, & a_1 = 1, & a_2 = 2, \\ A = 1, & a_1 = -2, & a_2 = 1, \\ A = 1, & a_1 = -1, & a_2 = 2, \end{array}$$

yielding 5 solutions of degree 2

$$2x^2, \qquad -x^2 - 3x, \qquad -x^2 + 3x, \qquad x^2 - x, \qquad x^2 + x.$$

For $n = 3$, equality (1) becomes

$$3 = Aa_1a_2a_3,$$

which has 2 solutions

$$\begin{array}{llll} A = -1, & a_1 = -1, & a_2 = 1, & a_3 = 3 \\ A = 1, & a_1 = -3, & a_2 = -1, & a_3 = 1 \end{array}$$

yielding 2 solutions of degree 3

$$-x^3 + 3x^2 + x, \qquad x^3 + 3x^2 - x.$$

For $n = 4$, equality (1) becomes

$$-4 = Aa_1a_2a_3a_4,$$

which has 1 solution

$$A = -1, \qquad a_1 = -1, \qquad a_2 = 1, \qquad a_3 = -2, \qquad a_4 = 2$$

yielding 1 solution of degree 4

$$-x^4 + 5x^2.$$

□

Problem 8. Determine all polynomials P(x) such that, for all real numbers x,

$$x \cdot P(x-1) = (x-3) \cdot P(x)$$

Solution. Setting $x = 0$ in

$$x \cdot P(x-1) = (x-3) \cdot P(x), \tag{2}$$

we obtain $0 = -3P(0)$, implying $P(0) = 0$. Setting $x = 1$ in (2) and using that $P(0) = 0$, we obtain $0 = -2P(1)$, implying $P(1) = 0$. Setting $x = 2$ in (2) and using that $P(1) = 0$, we obtain $0 = -P(2)$, implying $P(2) = 0$. Therefore, for all real numbers x,

$$P(x) = x(x-1)(x-2)Q(x),$$

for some polynomial $Q(x)$. Setting $P(x) = x(x-1)(x-2)Q(x)$ in (2) we obtain, for all real numbers x,

$$x(x-1)(x-2)(x-3)Q(x-1) = x(x-1)(x-2)(x-3)Q(x),$$

and this implies that

$$Q(x-1) = Q(x).$$

In particular, this means that $Q(x)$ has the same value at all integers, and this is possible only for constant polynomials. Therefore $Q(x) = A$, for some constant A and every polynomial that satisfies (2) must have the form $P(x) = Ax(x-1)(x-2)$. Direct check verifies that every polynomial of this form satisfies (2). Therefore

$$P(x) = Ax(x-1)(x-2),$$

for some constant A.

□

Problem 9. Show that if $P(x)$ is a polynomial with real coefficients, which has both positive and negative zeros, then $P(P(x))$ has at least one real zero.

Solution. Let $a < 0 < b$ and $P(a) = P(b) = 0$. Since the polynomial $P(x)$ achieves the value 0, the graph of the polynomial function $y = P(x)$ must intersect at least one of the horizontal lines $y = a$ and $y = b$. In the former case, $P(x') = a$ for some real number x', which yields $P(P(x')) = P(a) = 0$, and in the latter, $P(x'') = b$ for some real number x'', which yields $P(P(x'')) = P(b) = 0$. □

Problem 10. Determine all polynomials $P(x)$ such that, for all real numbers x and y

$$(x-y)P(x+y) - (x+y)P(x-y) = 4xy(x^2 - y^2). \tag{3}$$

Solution. Set

$$x - y = u \qquad \text{and} \qquad x + y = v.$$

As x and y range over all pars of real numbers, u and v also range over all pairs of real numbers. Indeed, the pair of values u' and v' is obtained by setting $x = \frac{u'+v'}{2}$ and $y = \frac{v'-u'}{2}$. Therefore, by setting $x - y = u$ and $x + y = v$ in (3), we obtain that

$$uP(v) - vP(u) = (v^2 - u^2)uv,$$

for all real numbers u and v. After dividing by uv, for $u \neq 0$ and $v \neq 0$,

$$\frac{P(v)}{v} - v^2 = \frac{P(u)}{u} - u^2.$$

The last identity implies that there exists a constant C such that, for $x \neq 0$,

$$\frac{P(x)}{x} - x^2 = C,$$

and therefore $P(x) = x^3 + Cx$, for $x \neq 0$. Since the polynomials $P(x)$ and $x^3 + Cx$ agree on infinitely many values, they must agree everywhere. Thus, for all x,

$$P(x) = x^3 + Cx.$$

□

Problem 11. Suppose the polynomial $P(x)$ has real coefficients and satisfies the equation

$$P(x)P(x+1) = P(x^2+x+1)\,.$$

Show that $P(x) = P(-x)$.

Solution. First observe that the polynomial $P(x)$ has no real roots. For if x is real and $P(x) = 0$, then $P(x^2+x+1) = 0$. Since $x^2+x+1 > x$, we must have an infinite number of real roots, which is impossible for a polynomial, which is not identically zero. In which case we have $P(x) = P(-x)$.

Replacing x with $-x-1$ in the equation gives the following

$$P(-x-1)P(-x) = P((-x-1)^2 + (-x-1) + 1) = p(x^2+x+1) = P(x)P(x+1)\,.$$

That is,

$$P(-x-1)P(-x) = P(x)P(x+1)\,.$$

Setting $x = 0$ in this last equation we have $P(-1)P(0) = P(0)P(1)$. Since $P(0) \neq 0$, we must have $P(1) = P(-1) \neq 0$. Similarly for any integer k we have $P(-k) = P(k)$. Since P is a polynomial we must have $P(-x) = P(x)$ for all x. $\square$