2013 Power Team Rules

(1) Each power team entry must have a cover sheet (typed). The cover sheet must contain the name of the school, the coach's name, the team name, and the names of each team member. For example, if a school has 3 power team entries, then there should be three different team names. The following is an example of an acceptable title page:

2013 Power Team Entry
XXXX School, Team 1
Coach: Ms. Wizard

Team Members:
Jane Doe
John Smith
etc.

(2) Team participants are not allowed to consult with anyone but their team mates. Participants are not allowed to look on the web for any information regarding the power team exam, nor are they allowed to search books or other reference materials.

(3) Team entries are expected to be neat and legible. If not, they face the possibility of being disqualified by the judges.

(4) All hand delivered submissions are due by 9:15 am on the day of the contest. FAXed submissions will be accepted provided they are received no later than 8:30 AM. on the day of the contest. FAXed solutions should be sent to 979-862-4190, attention Mike Stecher.

POWER TEAM 2013

Texas A&M University High School Math Contest

November 16, 2013

In working the problems below you may find the following lemma, which is due to Gauss, helpful.

Lemma 1. *Let $P(x) = Q(x)R(x)$ be a monic polynomial with integer coefficients. If Q and R are both polynomials with Q a monic polynomial with integer coefficients, then R is also a monic polynomial with integer coefficients. Be sure to mention when you use this fact.*

The word monic means that the coefficient of the highest power of x is 1. The word zero or root when applied to a polynomial means a number a such that $P(a) = 0$.

Problem 1. Let $P(x)$ be a polynomial with integer coefficients. Show that if a is an integer, then $P(x) - P(a) = (x - a)Q(x)$, where $Q(x)$ is a polynomial with integer coefficients.

Problem 2. Let

$$P(x) = x^5 + 10x^4 + 50x^3 + a_2x^2 + a_1x + a_0,$$

where a_0, a_1 and a_2 are some real numbers. Show that the polynomial $P(x)$ cannot have 5 real zeros.

Problem 3. Let $P(x)$ be a polynomial of degree 2013 with integer coefficients, such that $P(1) = P(2) = 17$. Show that for no integer a we may have $P(a) = 34$.

Problem 4. Show that there is no polynomial with integer coefficients such that there are three distinct integers a, b, and c such that

$$P(a) = b, \qquad P(b) = c, \qquad P(c) = a.$$

Problem 5. Let $n = 3^{2013}$ and $m = 2^{2013}$. Consider the coefficient in front of x^m in the polynomials

$$P(x) = (1 - x^2 + x^3)^n \qquad \text{and} \qquad Q(x) = (1 + x^2 - x^3)^n.$$

Which one is larger?

Problem 6. (a) Let

$$P(x) = (x - a_1)(x - a_2) \cdots (x - a_{2012}) - 1$$

where $a_1, a_2, \ldots, a_{2012}$ are distinct integers. Show that $P(x)$ cannot be written as a product

$$P(x) = Q(x)R(x)$$

of two nonconstant polynomials $Q(x)$ and $R(x)$ with integer coefficients.

(b) Let

$$P(x) = (x - a_1)(x - a_2) \cdots (x - a_{2013}) + 1$$

where $a_1, a_2, \ldots, a_{2013}$ are distinct integers. Show that $P(x)$ cannot be written as a product

$$P(x) = Q(x)R(x)$$

of two nonconstant polynomials $Q(x)$ and $R(x)$ with integer coefficients.

(c) Let $P(x)$ be a polynomial of degree 2013 such that there are distinct integers $a_1, a_2, \ldots, a_{2013}$ for which the value of the polynomial is 1 or -1 (1 for some of them and -1 for some of them). Show that $P(x)$ cannot be written as a product

$$P(x) = Q(x)R(x)$$

of two nonconstant polynomials $Q(x)$ and $R(x)$ with integer coefficients.

Problem 7. Find all polynomials $P(x)$ of degree n, $n \geq 2$, with integer coefficients, such that $P(0) = 0$ and there are n distinct integers $a_1, a_2, \ldots, a_n$ with $P(a_i) = n$, for $i = 1, 2, \ldots, n$.

Problem 8. Determine all polynomials P(x) such that, for all real numbers x,

$$x \cdot P(x - 1) = (x - 3) \cdot P(x)$$

Problem 9. Show that if $P(x)$ is a polynomial with real coefficients, which has both positive and negative zeros, then $P(P(x))$ has at least one real zero.

Problem 10. Determine all polynomials $P(x)$ such that, for all real numbers x and y

$$(x - y)P(x + y) - (x + y)P(x - y) = 4xy(x^2 - y^2). \tag{1}$$

Problem 11. Suppose the polynomial $P(x)$ has real coefficients and satisfies the equation

$$P(x)P(x + 1) = P(x^2 + x + 1)\,.$$

Show that $P(x) = P(-x)$.