AB EXAM SOLUTIONS
TEXAS A&M HIGH SCHOOL MATH CONTEST
NOVEMBER 16, 2013

1. Now $\dfrac{a+b}{2} = x$ and $\dfrac{b+x}{2} = \dfrac{a+b+1}{2}$, so $a+b = 2x$ and

$$\begin{aligned} b+x &= a+b+1 \\ x &= a+1 \\ 2a+2 &= a+b \\ a+2 &= b \\ a-b &= -2 \end{aligned}$$

2.

$$\begin{aligned} 2x^2+8xy+8y^2 &= 2(x^2+4xy+4y^2) \\ &= 2(x+2y)(x+2y) \\ &= 2(4)(4) \\ &= 32\ , \end{aligned}$$

the length of a side. So the perimeter is $4(32) = 128$.

3. $m+2n$ is the largest, so it is the length of the hypotenuse. Now

$$\begin{aligned} (m+2n)^2 &= (m+n)^2 + m^2 \\ m^2+4mn+4n^2 &= m^2+2nm+n^2+m^2 \\ m^2-2mn-3n^2 &= 0. \end{aligned}$$

Divide by n^2 to obtain

$$\begin{aligned} \left(\frac{m}{n}\right)^2 - 2\left(\frac{m}{n}\right) - 3 &= 0 \\ \left(\frac{m}{n}-3\right)\left(\frac{m}{n}+1\right) &= 0 \\ \frac{m}{n} &= 3\,. \end{aligned}$$

4. Each elephant has 4 knees, so

$$4E + 2M = 100\,.$$

Counting trunks we have

$$E + 3M = 100\,.$$

Solving the system gives $M = 30$ and $E = 10$. So $\dfrac{E}{M} = \dfrac{10}{30} = \dfrac{1}{3}$.

5. Let $x = 2$ to obtain $3g(2) + 2g(-1) = 13$ and let $x = -1$ to obtain $3g(-1) + 2g(2) = 7$. Solving

$$\begin{aligned} 3g(2) + 2g(-1) &= 13 \\ 2g(2) + 3g(-1) &= 7 \end{aligned}$$

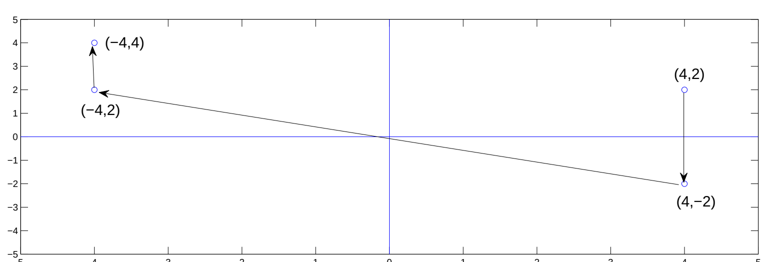
gives $g(2) = 5$ and $g(-1) = -1$.

6. Since p and q are roots then $p + q = 6$ and $pq = 2$. So

$$\frac{1}{p} + \frac{1}{q} = \frac{q+p}{qp} = \frac{6}{2} = 3\,.$$

7. Hasse walks 4 blocks and flips the coin 3 times. He will arrive at his starting point with either RRR or LLL. There are $2^3 = 8$ possibilities, RLR, RRL, LRL, etc. So the probability Hasse returns to the starting point is $\dfrac{2}{8} = \dfrac{1}{4}$.

8. See below:



9. Now $f(5w) = f(25w/5) = (25w)^2 + 25w + 3 = 625w^2 + 25w + 3$. We want

$$\begin{aligned} 625w^2 + 25w + 3 &= 2013 \\ 625w^2 + 25w - 2010 &= 0 \\ 125w^2 + 5w - 402 &= 0 \\ w^2 + \frac{1}{25}w - \frac{402}{125} &= 0\,. \end{aligned}$$

The sum of the roots in this quadratic is $-\dfrac{1}{25}$.

10. There are eight letters with I repeated 3 times and N two times.

$$\frac{8!}{3!2!} = 8 \cdot 7 \cdot 5 \cdot 4 \cdot 3 = 3360\,.$$

11.

$$\begin{aligned}
\sqrt{4-x} + \sqrt{4+x} &= 2x \\
(\sqrt{4-x} + \sqrt{4+x})^2 &= (2x)^2 \\
4 - x + 2\sqrt{4-x}\sqrt{4+x} + 4 + x &= 4x^2 \\
2\sqrt{4-x}\sqrt{4+x} &= 4x^2 - 8 \\
\sqrt{4-x}\sqrt{4+x} &= 2x^2 - 4 \\
(\sqrt{4-x}\sqrt{4+x})^2 &= (2x^2 - 4)^2 \\
(4-x)(4+x) &= 4x^4 - 16x^2 + 16 \\
16 - x^2 &= 4x^4 - 16x^2 + 16 \\
4x^4 - 15x^2 &= 0 \\
x^2(4x^2 - 15) &= 0,
\end{aligned}$$

giving $x = 0, x = \pm\frac{\sqrt{15}}{2}$. Substituting $x = 0$ and $x = -\frac{\sqrt{15}}{2}$ back into the equation shows they are not solutions.

12. Since $2013 = 3(671)$, then 671 numbers are divisible by 3. Also $2013 = 7(287) + 4$ means 287 are divisible by 7, and $2013 = 21(95) + 18$ means 95 are divisible by both 3 and 7. We conclude that $287 + 671 - 95 = 863$ are divisible be either 3 or 7 (or perhaps both). So $2013 - 863 = 1150$ are divisible by neither 3 nor 7.

13. Let x have ones digit a and tens digits b. Then $x = 10b + a$ and

$$\begin{aligned}
4(10b + a) - 5(10a + b) &= 4 \\
40b + 4a - 50a - 5b &= 4 \\
35b - 46a &= 4 \\
35b &= 2(2 + 23a)
\end{aligned}$$

Now 5 and 7 must divide $2 + 23a$ with $a \in \{1, 2, \ldots, 9\}$. So $a = 6$ and from this we obtain $b = 8$. Thus, $x = 86$.

14. Now

$$9 = \left(x + \frac{1}{x}\right)^2 = x^2 + 2 + \frac{1}{x^2}$$

so

$$
\begin{aligned}
x^2 + \frac{1}{x^2} &= 7 \\
\left(x^2 + \frac{1}{x^2}\right)^2 &= 49 \\
x^4 + 2 + \frac{1}{x^4} &= 49 \\
x^4 + \frac{1}{x^4} &= 47\,.
\end{aligned}
$$

15. Consider

$$
\begin{array}{r}
95d \\
-\underline{d59}
\end{array}
$$

In the units position we necessarily have $d < 9$ and in the subtraction process one needs to "borrow" a one from the tens position. This gives

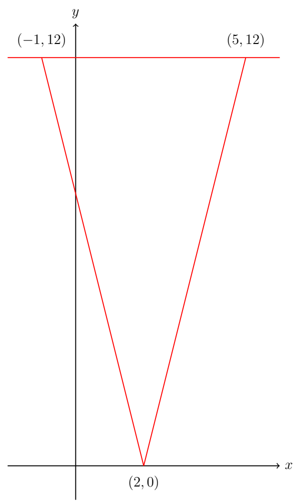
$$
\begin{array}{r}
95d \\
-\underline{d59} \\
9\,
\end{array}
$$

Again one needs to "borrow" a 1 from the hundreds position and either $8 - d = 5$ or $1d - 9 = 5$. The latter works with $d = 4$:

$$
\begin{array}{r}
954 \\
-\underline{459} \\
495
\end{array}
$$

16. Now $4^{300} = 2^{600} > 2^{500}$ so keep 2^{500}. Also, $2^{500} = (2^5)^{100} = 32^{100} < 81^{100} = (3^4)^{100} = 3^{400}$, so again keep 2^{500}. Finally $2^{500} = 32^{100} > 25^{100} = (5^2)^{100} = 5^{200}$ and 5^{200} is smallest.

17. The graph of $y = |4x - 8|$ is as picture



The region in question is the triangular region bounded by the above graph and the horizontal line $y = 12$. The vertices of this triangle are $(-1, 12), (5, 12), (2, 0)$; an isosceles triangle. The length of the base is $5 - (-1) = 6$ and height is 12. So the area is $(1/2)(6)(12) = 36$.

18.

$$\begin{aligned}\frac{n^2 - 38}{n+1} &= \frac{n^2 - 1 + 1 - 38}{n+1} \\ &= \frac{n^2-1}{n+1} - \frac{37}{n+1} \\ &= n - 1 - \frac{37}{n+1}\,.\end{aligned}$$

The largest n such that $\dfrac{37}{n+1}$ is an integer is $n + 1 = 37$ or $n = 36$.

19. The pattern repeats with a "fundamental period" of length 11. Since $2013 = 183 \cdot 11 + 0$, then the letter in position 2013 will be the same as that in the 11^{th} position, namely R.

20. Since $2013 = 7(287) + 4$, then

$$\begin{aligned} 2013 &\equiv 4 \bmod 7 \\ (2013)^2 &\equiv 16 \equiv 2 \bmod 7 \\ (2013)^3 &\equiv 8 \equiv 1 \bmod 7 \\ (2013)^4 &\equiv 4 \bmod 7 \\ &\vdots \end{aligned}$$

Note that $4 + 2 + 1 \equiv 0 \bmod 7$. So

$$S \equiv 1 + (4+2+1) + (4+2+1) + \cdots \qquad (\bmod 7)\,.$$

We have

$$\begin{aligned} (2013)^{2013} &\equiv (4)^{2013} \bmod 7 \\ &\equiv (4)^{(3)(671)} \bmod 7 \\ &\equiv (1)^{671} \bmod 7 \\ &\equiv 1 \bmod 7 \end{aligned}$$

and

$$\begin{aligned} S &\equiv 1 + (4+2+1) + (4+2+1) + \cdots \qquad (\bmod 7) \\ &\equiv 1 \bmod 7\,. \end{aligned}$$

21. There are 13 letters with N and U repeated. So there are $\dfrac{(13)!}{2!2!}$ distinct arrangements. The vowels are A,E,I,O,U,U. (6 in all). To form an arrangement with the vowels in alphabetical order, we first form an arrangement of the non-vowels ($7!/2$ different ways) and then insert the six vowels in alphabetical order. This latter is done by selecting a subset of 6 elements from $\{1, 2, \ldots, 13\}$ to obtain positions for the 6 vowels. (Example: $\{2, 5, 9, 10, 12, 13\}$ means A is in position 2, E is in position 5, I is in position 9, etc.) So our arrangment can be found in $\dfrac{7!}{2}\dbinom{13}{6}$ different ways. The probability is

$$\frac{\dfrac{7!}{2}\dbinom{13}{6}}{\dfrac{(13)!}{2 \cdot 2}} = \frac{2 \cdot 7!}{(13)!}\frac{(13)!}{7! \cdot 6!} = \frac{2}{6!} = \frac{1}{6 \cdot 5 \cdot 4 \cdot 3} = \frac{1}{360}\,.$$