BC EXAM
Texas A&M High School Math Contest. Solutions
November 2013

Directions: If units are involved, include them in your answer.

1. An aquarium has a rectangular base that measures $100cm$ by $40cm$ and has a height of $50cm$. It is filled with water to a height of $40cm$. A brick with a rectangular base that measures $40cm$ by $20cm$ and a height of $10cm$ is placed in the aquarium. By how many centimeters does the water rise?

   Solution. The brick has a volume of $40 \cdot 20 \cdot 10 = 800cm^3$. Suppose that after the brick is places in the tank, the water level rises by h centimeters. Then the additional volume occupied in the aquarium is $100 \cdot 40 \cdot h = 4000h$ cubic centimeters. Since this must be the same as the volume of the brick, we have $8000 = 4000h$ and $h = 2$ centimeters.

   Answer: $2cm$

2. How many non-congruent triangles with perimeter 7 have integer side lengths?

   Solution. The longest side cannot be greater than 3, since otherwise the remaining two sides would not be long enough to form a triangle. The only possible triangles have side lengths $1-3-3$ or $2-2-3$.

   Answer: 2

3. Let x and y be two-digit integers such that y is obtained by reversing the digits of x. The integers x and y satisfy $x^2 - y^2 = m^2$ for some positive integer m. What is $x + y + m$?

   Solution. By the given conditions, it follows that $x > y$. Let $x = 10a + b$ and $y = 10b + a$, where $a > b$. Then
   $$m^2 = x^2 - y^2 = (10a + b)^2 - (10b + a)^2 = 99a^2 - 99b^2 = 99(a^2 - b^2).$$
   Since $99(a^2 - b^2)$ must be a perfect square,
   $$a^2 - b^2 = (a + b)(a - b) = 11k^2$$
   for some positive integer k. Because a and b are distinct digits, we have $a-b \le 9-1 = 8$ and $a+b \le 9+8 = 17$. It follows that $a + b = 11$, $a - b = k^2$, and either $k = 1$ or $k = 2$.

   If $k = 2$, then $(a, b) = (7.5, 3.5)$, which is impossible. Thus $k = 1$ and $(a, b) = (6, 5)$. This gives $x = 65$, $y = 56$, $m = 33$, and $x + y + m = 154$.

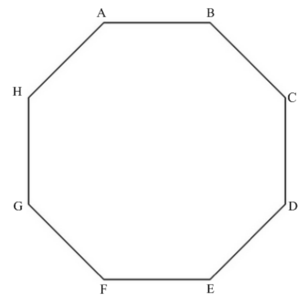
   Answer: 154

4. A two-digit integer x is to be chosen. If all choices are equally likely, what is the probability that at least one digit of x is 7?

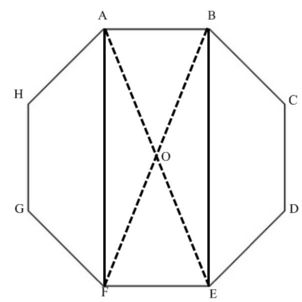
   Solution. There are 90 possible choices for x. Ten of these have a tens digit of 7, and nine have a units digits of 7. Because 77 has been counted twice, there are $10 + 9 - 1 = 18$ choices of x for which at least one digit is a 7. Hence, the probability is $\frac{18}{90} = 0.2$

   Answer: 0.2 (or anything equivalent, like 1/5 etc)

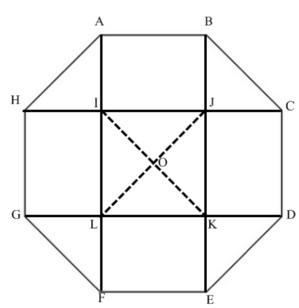
5. A regular octagon $ABCDEFGH$ has an area of one square unit. What is the area of rectangle $ABEF$?



Solution: *Way one* Draw the diagonals AE and BF and let O be the points of their intersection. Since the octagon is regular, the area of the triangle AOB is equal to $\frac{1}{8}$. Then the area of the rectangle $ABEF$ is four times of the area of the triangle AOB, i.e. it is equal to $\frac{1}{2}$.



Way two. The answer follows from the following partition of the octagon (see the picture below): the whole octagon is equal to the union of eight congruent small triangles and four congruent small rectangles, while the rectangle $ABEF$ is the union of four of those small triangles and two of those small rectangles.



Answer: $\dfrac{1}{2}$.

6. Find a positive four-digit integer which has the decimal representation $(abba)_{10}$ and it is a perfect cube.

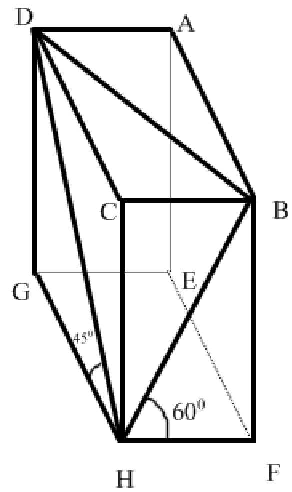
   Solution. Let $n = (abba)_{10}$. We have

$$(abba)_{10} = 1000a + 100b + 10b + a = 1001a + 110b = 11(91a + 10b).$$

   Therefor n is divisible by 11. Since n is also a perfect cube, we conclude that $n = 11^3k^3 = 1331k^3$ for some integer k. On the other hand $1331k^3 > 10000$ for $k > 1$. Thus, $k = 1$ and $n = 1331$.

   Answer: 1331

7. In the rectangular solid shown, we have $\angle DHG = 45°$ and $\angle FHB = 60°$. What is the cosine of $\angle BHD$?



Solution. Since no dimensions have been specified, we can assign a value to one of them and determine the others relative to that dimension. Let $GH = 1$. Since $\angle GHD = 45°$, this implies that

$$1 = GH = DG = DC = CH = BF$$

and $DH = \sqrt{2}$. In addition, since $\triangle HFB$ is $30 - 60 - 90°$ with its longest leg $BF = 1$, we have

$$BH = BD = \frac{2\sqrt{3}}{3}$$

and

$$BC = HF = \frac{1}{2}BH = \frac{\sqrt{3}}{3}.$$

Applying the Law of Cosines to $\triangle BHD$ gives

$$BD^2 = DH^2 + BH^2 - 2 \cdot DH \cdot BH \cos \angle BHD,$$

so

$$\cos \angle BHD = \frac{1}{2 \cdot DH \cdot BH}(DH^2 + BH^2 - BD^2) = \frac{1}{2 \cdot \sqrt{2} \cdot \frac{2\sqrt{3}}{3}}((\sqrt{2})^2 + (\frac{2\sqrt{3}}{3})^2 - (\frac{2\sqrt{3}}{3})^2) = \frac{\sqrt{6}}{4}.$$

Answer: $\frac{\sqrt{6}}{4}$

8. If positive integers x and y satisfy the equation $xy = 3(x + y) - 5$ and $x \neq y$, find $x + y$.

   Solution. We have $-5 = xy - 3(x + y) = (x - 3)(y - 3) - 9$. Hence, $(x - 3)(y - 3) = 4$. Taking into account that $x \neq y$, we get that either $x - 3 = 1, y - 3 = 4$, or $x - 3 = 4, y - 3 = 1$. In both cases $x + y = 11$.

   Answer: 11

9. Find the value of the constant k for which the graphs $2y + x + 3 = 0$ and $3y + kx + 2 = 0$ are perpendicular.

   Solution. The slopes of the given lines, $2y + x + 3 = 0$ and $3y + kx + 2 = 0$, are $m_1 = -1/2$ and $m_2 = -k/3$, respectively. These lines are perpendicular if $m_1 \cdot m_2 = -1$, which implies $k = -6$.

   Answer: -6

10. What is the sum of the digits in the number $10^{55} - 55$?

Solution. 10^{55} has 55 zero digits and

$$10^{55} - 55 = \underbrace{9\ldots9}_{53\ digits}45.$$

The sum of the digits is $53 \cdot 9 + 4 + 5 = 9 \cdot 54 = 486$.

Answer: 486

11. Let a, b, and c be positive integers satisfying the following system of equations:

$$\begin{cases} 7a^2 - 3b^2 + 4c^2 = & 8 \\ 16a^2 - 7b^2 + 9c^2 = & -3 \end{cases}$$

Find $a^2 + b^2 + c^2$.

Solution. Adding the first equation multiplied by 7 and the second equation multiplied by -3, we get

$$a^2 + c^2 = 65. \tag{1}$$

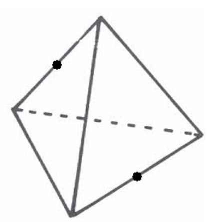
Since a and c are positive integers, they must be elements from the set $\{1,2,3,4,5,6,7,8\}$. It is easy to figure out that the only possible pairs (a, c) satisfying (1) are $(1, 8), (8, 1), (4, 7), (7, 4)$. In possible values of c are 1, 8, 7, or 4 Adding the first equation multiplied by 16 and the second equation multiplied by -7 we get

$$b^2 + c^2 = 149. \tag{2}$$

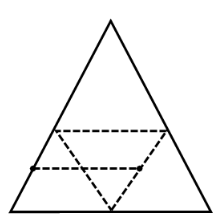
Note that $149 - 1 = 148$, $149 - 64 = 85$, and $149 - 16 = 133$ are not perfect squares, but $149 - 49 = 100$ is a perfect square. Therefore $c = 7 \Rightarrow a = 4 \Rightarrow b = 10$, i.e. the only solution of our system is $(a, b, c) = (4, 10, 7)$, which implies that $a^2 + b^2 + c^2 = 165$.

Answer: 165

12. An insect lives on the surface of a regular tetrahedron with edges of length 1. It wishes to travel on the surface of the tetrahedron from the midpoint of one edge to the midpoint of the opposite edge (see the picture). What is the length of the shortest such trip (Note that two edges of a tetrahedron are *opposite* if they have no common endpoint).



Solution. Unfold the tetrahedron onto a plane. The two opposite edge midpoint become midpoints of opposite sides of a rhombus with sides of length 1, so the shortest path between them on the plane has length one. Folding the triangle back to a tetrahedron does not change the length of the path and it remains the shortest one.



Answer: 1.

13. Find the following sum $1 + 4 + 7 + \ldots + 94 + 97 + 100$.

    Solution. Let $S = 1 + 4 + 7 + \ldots + 94 + 97 + 100$, then $S = 100 + 97 + 94 + \ldots + 7 + 4 + 1$. So, $2S = (1 + 100) + (4 + 97) + (7 + 94) + \ldots + (94 + 7) + (97 + 4) + (100 + 1)$, and the number of pairs is $(100 + 2) : 3 = 34$. Hence, $S = \dfrac{34 \cdot 101}{2} = 1717$.

    Answer: 1717

14. Calculate
$$\frac{8 + 222 \cdot 444 \cdot 888 + 444 \cdot 888 \cdot 1776}{2 \cdot 4 \cdot 8 + 444 \cdot 888 \cdot 1776 + 888 \cdot 1776 \cdot 3552}$$

    Solution.
$$\frac{8 + 222 \cdot 444 \cdot 888 + 444 \cdot 888 \cdot 1776}{2 \cdot 4 \cdot 8 + 444 \cdot 888 \cdot 1776 + 888 \cdot 1776 \cdot 3552} = \frac{1 \cdot 2 \cdot 4 \cdot (1 + 222^3 + 444^3)}{2 \cdot 4 \cdot 8 \cdot (1 + 222^3 + 444^3)} = \frac{1}{8}.$$

    Answer: $\frac{1}{8}$

15. Given p and q are positive integers satisfying the following equality $p! + 12 = q^2$, find $p + q$. (Recall that $p! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (p - 1) \cdot p$.)

    Solution. First note that if $p \geq 5$ then $p!$ is proportional to 2 and to 5, i.e. $p!$ is divisible by 10 and thus the number $p! + 12$ has 2 as its last digit. But such a number cannot be a perfect square of an integer. It remains to consider cases $p = 1, 2, 3, 4$. It is easy to conclude that only the pair $(p, q) = (4, 6)$ satisfies the given equation. Finally, $p + q = 10$.

    Answer: 10

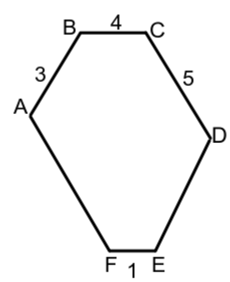
16. The sides of a triangle satisfy the following inequalities:
$$a \leq 5 \leq b \leq 6 \leq c \leq 8.$$

    Find the maximum possible area of such triangle.

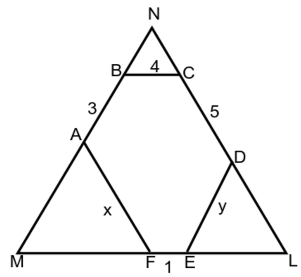
    Solution. Let γ be the angle between a and b. Then the area of triangle can be found by the formula: $A = \frac{1}{2}ab \sin \gamma$. Note that $\sin \gamma \leq 1$ and equality is attained when γ is right angle. Hence, $A \leq \frac{1}{2}ab \leq \frac{1}{2} \cdot 5 \cdot 6 = 15$, because the maximal values for a and b are 5 and 6, respectively. In this case the hypotenuse c is equal $\sqrt{61}$ and $6 < \sqrt{61} < 8$.

    Answer: 15

17. Each angle of the hexagon $ABCDEF$ is $120°$. Find $AF + DE$ if $AB = 3$, $BC = 4$, $CD = 5$ and $EF = 1$.



Solution. We extend the sides AB, CD and EF to their pairwise intersection of the points M, N and L (see the picture below).



Then each of the triangles MAF, BNC and DLE contains two angles that are $60°$, so these triangles are equilateral. Consequently, the triangle MNL is also equilateral. Then $AF+3+4 = 4+5+DE = DE+1+AF$, where $AF = 8$, $DE = 6$. Finally, $AF + DE = 14$.

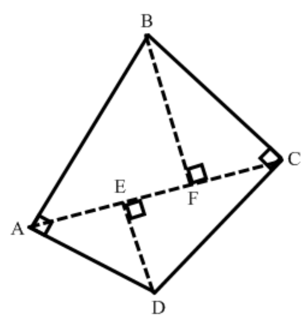
Answer: 14

18. Let ♠(n) denote the sum of the digits of the positive integer n. For example, ♠$(5) = 5$ and ♠$(8123) = 8 + 1 + 2 + 3 = 14$. For how many two-digit values of n is ♠(♠$(n)) = 3$?

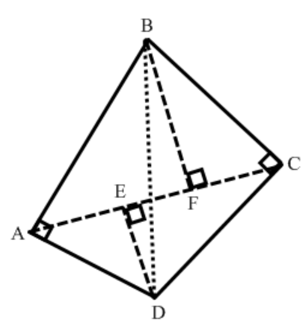
    Solution. Note that the maximum possible value for ♠(n) when n is a two-digit integer is $9 + 9 = 18$. Let $p =$ ♠(n). In order to have ♠$(p) = 3$, we must have p as 3,12, or 21. However $p = 21 > 18$ does not give valid solutions for n. If $3 = p =$ ♠(n), then $n = 12, 21$, or 30, and if $12 = p =$ ♠(n), then $n = 39, 93, 48, 84, 57, 75$, or 66. This gives 10 total possibilities for n.

    Answer: 10.

19. The quadrilateral $ABCD$ has right angles at A and at C. Points E and F are on AC with DE and BF perpendicular to AC. In addition, $AE = 3$, $DE = 5$, and $CE = 7$. What is BF?



Solution. First construct the diagonal BD.



Since $\angle BAD$ is a right angle, $\angle BAE + \angle EAD = 90°$. But the triangle AFB is a right triangle, so $\angle BAE + \angle ABF = 90°$ as well. Thus $\angle EAD = \angle ABF$, and the right triangle AED is similar to the right triangle AFB. Hence,

$$\frac{AF}{BF} = \frac{DE}{AE},$$

so

$$AF = \frac{DE}{AE} BF = \frac{5}{3} BF.$$

Similarly, $\angle CBF = \angle DCE$, and the right triangle CFB is similar to the right triangle DEC. So

$$\frac{CF}{BF} = \frac{DE}{EC},$$

so

$$CF = \frac{DE}{EC} BF = \frac{5}{7} BF.$$

Thus

$$AE + EC = 3 + 7 = 10 = AF + FC = \frac{5}{3} BF + \frac{5}{7} BF = \frac{50}{21} BF$$

and $BF = 21/5$.

Answer: 21/5

20. Three cubes having volumes 1, 8, and 27 are glued together at their faces. What is the smallest possible surface area that the resulting polyhedron can have?

Solution. The total surface area of the three cubes before they have been placed together is

$$6 \cdot (3 \times 3 + 2 \times 2 + 1 \times 1) = 84.$$

Placing the cube with volume 8 on one corner of the top face of the cube with volume 27 removes $2 \times 2 = 4$ square units of the surface area from each cube. This reduces the surface area to $84 - 2 \times 4 = 76$. Placing the unit cube so that one of its faces adjoins the cube with volume 27 and another adjoins the cube with volume 8 reduces the surface area an additional 4 square units, leaving the minimal surface area of $84 - 2 \cdot 4 - 4 \cdot 1 = 72$ square units.

Answer: 72