CD Exam, Solutions
Texas A&M High School Math Contest
Nov. 16, 2013

1. Let A and B be the ages of Ann and Barbara, respectively. Then $B-A/2$ years ago Barbara was half as old as Ann is. Ann was then $A-(B-A/2)=\frac{3}{2}A-B$ years old. Barbara was that old $B-\left(\frac{3}{2}A-B\right)=2B-\frac{3}{2}A$ years ago. Ann was then $A-\left(2B-\frac{3}{2}A\right)=\frac{5}{2}A-2B$ years old. Therefore

$$B=\frac{5}{2}A-2B,$$

or $3B=\frac{5}{2}A$. We have $A+B=44$, hence $3(44-A)=\frac{5}{2}A$, so $132=\frac{5}{2}A+3A=\frac{11}{2}A$, hence $A=24$, and $B=20$.

2. Radius of the inscribed circle is 1/3 of the height, i.e., $s\frac{\sqrt{3}}{6}$. Then the side of the square is $\sqrt{2}$ times the radius, i.e., $s\frac{\sqrt{6}}{6}=s\frac{1}{\sqrt{6}}$. The area of the square is hence $s^2/6$.

3. Consider the following cases

(a) $x\le -1$;

(b) $-1<x\le 0$;

(c) $0<x\le 1$;

(d) $1<x\le 2$;

(e) $2<x$.

Then the equation is equivalent to

(a) $(-x-1)+x+3(-x+1)-2(-x+2)=x+2$, $-x-2=x+2$, hence $x=-2$.

(b) $(x+1)+x+3(-x+1)-2(-x+2)=x+2$, $x=x+2$, contradiction.

(c) $(x+1)-x+3(-x+1)-2(-x+2)=x+2$, $-x=x+2$, $x=-1$, which does not satisfy the condition $0<x\le 1$.

(d) $(x+1)-x+3(x-1)-2(-x+2)=x+2$, $5x-6=x+2$, $x=2$.

(e) $(x+1)-x+3(x-1)-2(x-2)=x+2$, $x+2=x+2$ is satisfied for all $x>2$.

Answer: $x=-2$ and all $x\ge 2$.

4. We have $\frac{1}{(2n-1)(2n+1)}=\frac{1}{2}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)$, hence

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{2011\cdot 2013}=\frac{1}{2}\left(\frac{1}{1}-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+\cdots+\frac{1}{2011}-\frac{1}{2013}\right)=\frac{1}{2}\left(1-\frac{1}{2013}\right)=\frac{1006}{2013}.$$

5. It is the same as the last digit of the product of the last digits of these numbers, which is $(2\cdot4\cdot6\cdot8)^{10}=384^{10}$, hence it is the same as the last digit of $4^{10}=1024^2$, hence it is the same as the last digit of $4^2=16$, i.e., it is 6.

6. We consider the cases $x\le -2$, $-2\le x\le 1$, and $x\ge 1$. In the first case the inequality is equivalent to $1-x-2-x<5$, which is equivalent to $-2x<6$, or $x>-3$. This gives us the interval $-3<x\le -2$.

In the second case we get $1-x+x+2<5$, which is always true, hence the set $-2\le x\le 1$ belongs to the set of solutions.

In the third case we get $x-1+x+2<5$, i.e., $2x<4$, $x<2$. This gives us $1\le x<2$.

It follows that the set of solutions is the interval $(-3,2)$.

7. The 1's are on the places number $1, 1+2=3, 1+2+3=6, 1+2+3+4=10, \ldots$, i.e., on places of the form $\frac{n(n+1)}{2}$. Let us find the biggest number n such that $\frac{n(n+1)}{2} \leq 1234$. The inequality is equivalent to $n^2+n \leq 2468$. Roots of $x^2+x-2468$ are $\frac{-1\pm\sqrt{1+4\cdot 2468}}{2} = \frac{-1\pm\sqrt{9873}}{2}$. We have $99 < \sqrt{9873} < 100$, hence the positive root is between $\frac{-1+99}{2} = 49$ and $\frac{-1+100}{2} = 49.5$. It follows that in the first 1234 elements of the sequence we have 49 ones. Consequently, the sum is $2 \cdot 1234 - 49 = 2468 - 49 = 2419$.

8. The equation is equivalent to $(2k-1)x^2 - 8x + 6 = 0$. Its discriminant is

$$D = 64 - 4 \cdot (2k-1) \cdot 6 = 64 - 48k + 24 = 88 - 48k.$$

The equation has no real roots if and only if $88 - 48k < 0$, i.e., $k > \frac{88}{48} = \frac{11}{6} = 1 + \frac{5}{6}$. Hence, the answer is $k = 2$.

9. The region is the annular area between the circle and concentric circle of radius $\sqrt{1+4} = \sqrt{5}$. Its area is $5\pi - 4\pi = \pi$.

10. The equation is equivalent to $5^A 2^B = 200^C$, i.e., to $5^A 2^B = 5^{2C} 2^{3C}$, which is in turn equivalent to $A = 2C, B = 3C$. It follows that $\frac{A}{B} = \frac{2}{3}$, and since A and B can not have common factors, the only possibility is that $A = 2$ and $B = 3$, hence $C = 1$.

11. We have $x^3 = x^2 - x + 2$, hence $p^3+q^3+r^3 = p^2-p+2+q^2-q+2+r^2-r+2 = p^2+q^2+r^2-(p+q+r)+6$. We have $p+q+r = 1$, and $pq+qr+pr = 1$, hence $1 = (p+q+r)^2 = p^2+q^2+r^2+2(pq+qr+pr) = p^2+q^2+r^2+2$, hence $p^2+q^2+r^2 = -1$, so that

$$p^3 + q^3 + r^3 = -1 - 1 + 6 = 4.$$

12. Taking square of the equation, we get $2 - \sqrt{2+x} = x^2$, or $\sqrt{2+x} = 2 - x^2$. Squaring again, we get $2 + x = (2-x^2)^2$, i.e., $2 + x = 4 - 4x^2 + x^4$. We see that $x^4 - 4x^2 - x + 2 = 0$. We have

$$\begin{aligned} x^4 - 4x^2 - x + 2 &= x^2(x^2-4) - (x-2) = x^2(x-2)(x+2) - (x-2) = \\ &(x-2)(x^3+2x^2-1) = (x-2)(x^3+x^2+x^2-1) = \\ &(x-2)(x^2(x+1)+(x-1)(x+1)) = (x-2)(x+1)(x^2+x-1) \end{aligned}$$

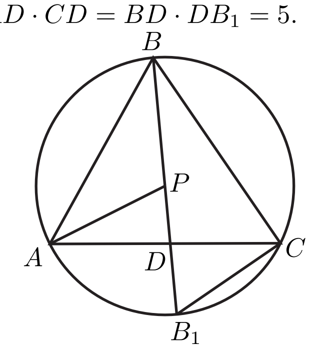
Roots of x^2+x-1 are $\frac{-1\pm\sqrt{1+4}}{2} = \frac{-1\pm\sqrt{5}}{2}$, hence roots of x^4-4x^2-x+2 are $2, -1, \frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}$.

Negative roots are not allowed, since $x = \sqrt{2-\sqrt{2+x}}$ is positive. The root $x = 2$ is also not allowed, since then $\sqrt{2-\sqrt{4}} = 0 \neq x$. Only $x = \frac{\sqrt{5}-1}{2}$ remains.

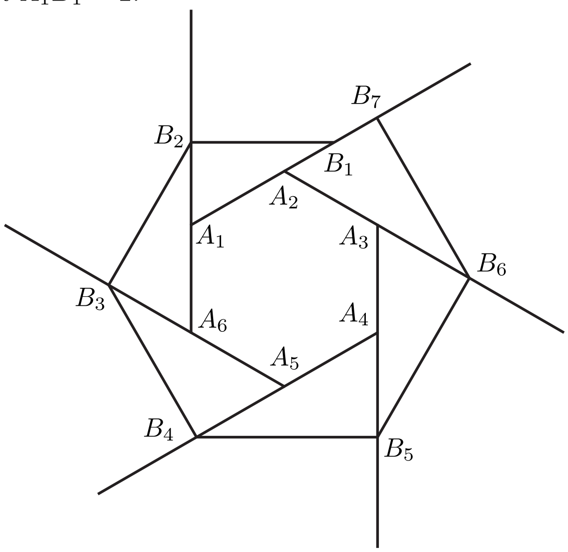
13. Subtracting the equations, we have $x^2 - y^2 + y - x = 0$, hence $(x-y)(x+y-1) = 0$. If $x = y$, then $x^2 + x = 3/4$, so $x = 1/2$ or $x = -3/2$, which gives solutions $(x,y) = (1/2, 1/2)$ and $(x,y) = (-3/2, -3/2)$.

If $x + y = 1$, then $y = 1 - x$, and $x^2 + 1 - x = 3/4$, $x^2 - x + 1/4 = 0$, which again gives the solution $(x,y) = (1/2, 1/2)$.

14. Draw the circle with center in P and radius AP. Then B is on the circle, since $PA = PB$. The vertex C is also on the circle, since $\angle APB$ is twice $\angle ACB$. Let B_1 be the second intersection of the line PB with the circle. Then $DB_1 = 1$, $BD = 5$. Triangles $\triangle BDA$ and $\triangle CDB_1$ are similar, hence $\frac{AD}{BD} = \frac{DB_1}{CD}$, so that $AD \cdot CD = BD \cdot DB_1 = 5$.

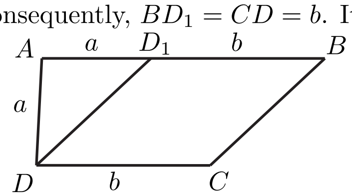


15. Length of each side of the hexagon is 1. Let $A_2B_1 = x$. Then $\triangle A_1B_2B_1$ is right triangle with $\angle B_2A_1B_1 = 60°$. It follows that $A_1B_2 = \frac{x+1}{2}$, hence $A_6B_2 = \frac{x+3}{2}$. By the same arguments, $A_6B_3 = \frac{x+3}{4}$, $A_5B_3 = \frac{x+7}{4}$, $A_5B_4 = \frac{x+7}{8}$, $A_4B_4 = \frac{x+15}{8}$, $A_4B_5 = \frac{x+15}{16}$, $A_3B_5 = \frac{x+31}{16}$, $A_3B_6 = \frac{x+31}{32}$, $A_2B_6 = \frac{x+63}{32}$, $A_2B_7 = \frac{x+63}{64}$. Since $B_7 = B_1$, we have $x = \frac{x+63}{64}$, hence $64x = x + 63$, so $63x = 63$, hence $x = 1$. It follows that $A_1B_1 = 2$.



16. We have to find the number of possible solutions of the equation $10d+25q = 1000$ in positive integers. It is equivalent to the equation $2d + 5q = 200$. We have that d must be divisible by 5 and satisfy $1 \le d < 100$, hence $1 \le d/5 \le 19$, and we have 19 possibilities for d. In each of these cases $q = \frac{200-2d}{5} = 40 - 2 \cdot \frac{d}{5}$ is a positive integer. Consequently, the answer is 19.

17. Draw the bisector of the angle D. Let D_1 be its intersection with AB. Since AB is parallel to CD, $\angle ADD_1 = \angle D_1DC = \angle AD_1D$, hence $\triangle ADD_1$ is isosceles, hence $AD_1 = AD = a$. Quadrilateral DD_1BC has two parallel sides and two equal opposite angles, $\angle D_1DC = \angle D_1BC$, hence it is a parallelogram. Consequently, $BD_1 = CD = b$. It follows that $AB = a + b$.



18. Let S be the area of the triangle. The area of the largest part is S minus the area of a similar triangle with similarity coefficient $9/10$. Its area is 0.9^2S. It follows that $S - 0.9^2S = 38$, or $(1 - 0.81)S = 38$, $0.19S = 38$, hence $S = 200$.

19. We have $P(0) = 0$, hence $P(1) = P(0^2+1) = (P(0))^2+1 = 1$, hence $P(2) = P(1^2+1) = P(1)^2+1 = 2$, hence $P(5) = P(2^2 + 1) = (P(2))^2 + 1 = 5$, $P(26) = P(5^2 + 1) = (P(5))^2 + 1 = 26$, etc. This way we get an infinite sequence a_n given by the relation $a_{n+1} = a_n^2 + 1$ such that $P(a_n) = a_n$. Since for two different polynomials $P(x)$ and $Q(x)$ the set of solutions of the equation $P(x) = Q(x)$ is finite, it follows that $P(x) = x$.

20. Either y or t is the biggest number among x, y, z, t. Let us assume that it is t. Then $x + y \le 2t$, hence z is either 1 or 2. If $z = 2$, then $x = y = t$, and the second equation is $2 + t = 2t$, which implies that $x = y = z = t = 2$.

If $z = 1$, then $1 + t = xy$ and $x + y = t$, which implies that $1 + x + y = xy$, hence $2 = 1 - x - y + xy$, i.e., $2 = (x - 1)(y - 1)$. Since we assume that $x \le y$, this implies that $x = 2$ and $y = 3$. Then $t = x + y = 5$. We get a solution $(x, y, z, t) = (2, 3, 1, 5)$.

If y is the biggest number, then the same arguments give us solutions $(x, y, z, t) = (2, 2, 2, 2)$ and $(1, 5, 2, 3)$.

Consequently, the solutions are $(2, 2, 2, 2)$, $(2, 3, 1, 5)$, and $(1, 5, 2, 3)$.

21. Multiplying the equation by 7!, we get

$$5 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7a_2 + 4 \cdot 5 \cdot 6 \cdot 7a_3 + 5 \cdot 6 \cdot 7a_4 + 6 \cdot 7a_5 + 7a_6 + a_7,$$

or

$$3600 = 7\,(6\,(5\,(4\,(3a_2 + a_3) + a_4) + a_5) + a_6) + a_7.$$

Since $0 \leq a_7 < 7$, it follows that a_7 is the remainder of division of 3600 by 7, hence $a_7 = 2$, and

$$514 = 6\,(5\,(4\,(3a_2 + a_3) + a_4) + a_5) + a_6.$$

By the same argument, a_6 is the remainder of division of 514 by 6, hence $a_6 = 4$, and

$$85 = 5\,(4\,(3a_2 + a_3) + a_4) + a_5.$$

We see that $a_5 = 0$, hence

$$17 = 4\,(3a_2 + a_3) + a_4,$$

$a_4 = 1$, and

$$4 = 3a_2 + a_3,$$

hence $a_3 = 1$ and $a_2 = 1$.

Answer: $(a_2, a_3, a_4, a_5, a_6, a_7) = (1, 1, 1, 0, 4, 2)$.