DE EXAM
Texas A&M High School Math Contest
November 2013

Directions: If units are involved, include them in your answer.

1. Solve the equation:
$$\max(x; 2-x) = \min(3x; 1+2x).$$
(Recall that $\max(a; b)$ is the greater of the two numbers a and b, and $\min(a; b)$ is the lesser of the two numbers a and b.)

2. Let a, b, and c be positive integers satisfying the following system of equations:
$$\begin{cases} 7a^2 - 3b^2 + 4c^2 = & 8 \\ 16a^2 - 7b^2 + 9c^2 = & -3 \end{cases}$$
Find $a^2 + b^2 + c^2$.

3. Find the following sum $1 + 4 + 7 + \ldots + 94 + 97 + 100$.

4. Calculate
$$\frac{8 + 222 \cdot 444 \cdot 888 + 444 \cdot 888 \cdot 1776}{2 \cdot 4 \cdot 8 + 444 \cdot 888 \cdot 1776 + 888 \cdot 1776 \cdot 3552}$$

5. Find m such that the following equations have at least one common root:
$$\begin{aligned} x^3 + mx + 1 &= 0, \\ x^4 + mx^2 + 1 &= 0. \end{aligned}$$

6. Given p and q are positive integers satisfying the following equality $p! + 12 = q^2$, find $p + q$. (Recall that $p! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (p-1) \cdot p$.)

7. Find the largest value a such that one of the roots of the following equation is greater than or equal to 1 and the other is less than or equal to 1:
$$(a^2 + a + 1)x^2 + (2a - 3)x + a - 5 = 0.$$

8. The sides of a triangle satisfy the following inequalities:
$$a \leq 5 \leq b \leq 6 \leq c \leq 8.$$
Find the maximum possible area of such triangle.

9. A function $f(x)$ satisfies $2f(x) + f(x^2 - 1) = 1$ for all real x. Find $f(-\sqrt{2})$.

10. How many solutions does the equation
$$\sqrt{x + 6 - 2x^2} \cdot \cos(\pi x) = 0$$
have?

11. Compute $S = \cos 36^\circ - \sin 18^\circ$.

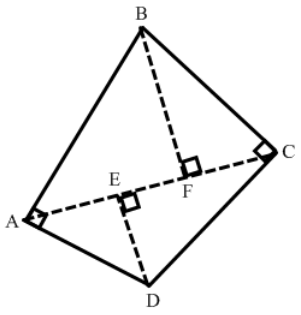
12. Given sequence $a_1 = 2$, $a_2 = 6$, $a_3 = 12$, $a_4 = 20$, $a_5 = 30$, $a_6 = 42$,.... Compute a_{2013}.

13. Each angle of the hexagon $ABCDEF$ is $120°$. Find $AF + DE$ if $AB = 3$, $BC = 4$, $CD = 5$ and $EF = 1$.

14. Let $\spadesuit(n)$ denote the sum of the digits of the positive integer n. For example, $\spadesuit(5) = 5$ and $\spadesuit(8123) = 8 + 1 + 2 + 3 = 14$. For how many two-digit values of n is $\spadesuit(\spadesuit(n)) = 3$?

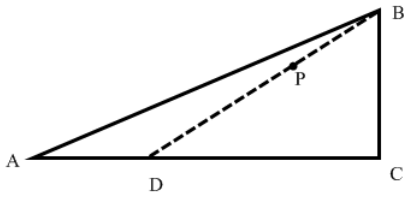
    Solution.

15. The quadrilateral $ABCD$ has right angles at A and at C. Points E and F are on AC with DE and BF perpendicular to AC. In addition, $AE = 3$, $DE = 5$, and $CE = 7$. What is BF?

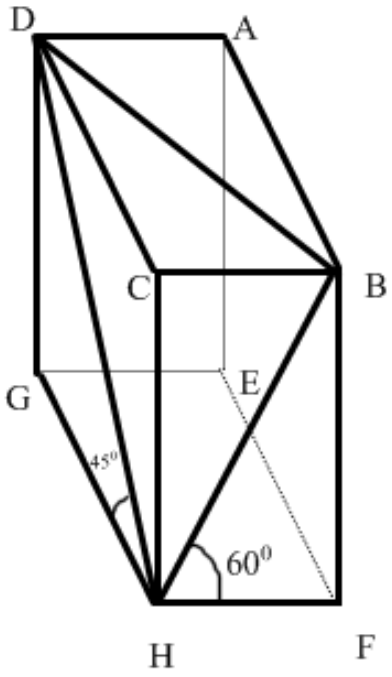


16. Three cubes having volumes 1, 8, and 27 are glued together at their faces. What is the smallest possible surface area that the resulting polyhedron can have?

17. Triangle ABC is a right triangle with $\angle ACB$ as its right angle, $\angle ABC = 60°$, and $AB = 10$. Point P is randomly chosen inside the triangle ABC, and the segment BP is extended to meet the side AC at D. What is the probability that $BD > 5\sqrt{2}$?



18. In the rectangular solid shown, we have $\angle DHG = 45°$ and $\angle FHB = 60°$. What is the cosine of $\angle BHD$?



19. Suppose that

$$\log_{2013}(\log_{2014}(\log_{2015}(\log_7 N))) = 2012.$$

How many different prime numbers are factors of N?

20. Given that $\sin x = 3\cos x$. What is $\sin x \cos x$?