DE EXAM
Texas A&M High School Math Contest
November 2013

Directions: If units are involved, include them in your answer.

1. Solve the equation:
$$\max(x; 2-x) = \min(3x; 1+2x).$$
(Recall that $\max(a;b)$ is the greater of the two numbers a and b, and $\min(a;b)$ is the lesser of the two numbers a and b.)

   Solution. First note that the inequalities $x \geq 2-x$ and $3x \geq 1+2x$ are both equivalent to $x \geq 1$. Hence,
$$\max(x; 2-x) = \begin{cases} x, & x \geq 1, \\ 2-x & x < 1, \end{cases}$$
and
$$\min(3x; 1+2x) = \begin{cases} 1+2x, & x \geq 1, \\ 3x & x < 1, \end{cases}.$$
Thus, the given equation is equivalent to $x = 1+2x$ (which is impossible) if $x \geq 1$; and it is equivalent to $2-x = 3x$ (i.e. $x = 0.5$) if $x < 1$.

   Answer: 0.5

2. Let a, b, and c be positive integers satisfying the following system of equations:
$$\begin{cases} 7a^2 - 3b^2 + 4c^2 = & 8 \\ 16a^2 - 7b^2 + 9c^2 = & -3 \end{cases}$$
Find $a^2 + b^2 + c^2$.

   Solution. Adding the first equation multiplied by 7 and the second equation multiplied by -3, we get
$$a^2 + c^2 = 65. \tag{1}$$
Since a and c are positive integers, they must be elements from the set $\{1,2,3,4,5,6,7,8\}$. It is easy to figure out that the only possible pairs (a, c) satisfying (1) are $(1,8), (8,1), (4,7), (7,4)$. Plugging in these pairs to the first given equation and taking into account that b is a positive integer, we conclude that the given system has a unique solution. Namely, $(a, b, c) = (4, 7, 10)$, which implies that $a^2 + b^2 + c^2 = 165$.

   Answer: 165

3. Find the following sum $1 + 4 + 7 + \ldots + 94 + 97 + 100$.

   Solution. Let $S = 1 + 4 + 7 + \ldots + 94 + 97 + 100$, then $S = 100 + 97 + 94 + \ldots + 7 + 4 + 1$. So, $2S = (1 + 100) + (4 + 97) + (7 + 94) + \ldots + (94 + 7) + (97 + 4) + (100 + 1)$, and the number of pairs is $(100 + 2) : 3 = 34$. Hence, $S = \dfrac{34 \cdot 101}{2} = 1717$.

   Answer: 1717

4. Calculate
$$\frac{8 + 222 \cdot 444 \cdot 888 + 444 \cdot 888 \cdot 1776}{2 \cdot 4 \cdot 8 + 444 \cdot 888 \cdot 1776 + 888 \cdot 1776 \cdot 3552}$$

   Solution.
$$\frac{8 + 222 \cdot 444 \cdot 888 + 444 \cdot 888 \cdot 1776}{2 \cdot 4 \cdot 8 + 444 \cdot 888 \cdot 1776 + 888 \cdot 1776 \cdot 3552} = \frac{1 \cdot 2 \cdot 4 \cdot (1 + 222^3 + 444^3)}{2 \cdot 4 \cdot 8 \cdot (1 + 222^3 + 444^3)} = \frac{1}{8}.$$

   Answer: $\frac{1}{8}$

5. Find m such that the following equations have at least one common root:

$$\begin{aligned} x^3 + mx + 1 &= 0, \\ x^4 + mx^2 + 1 &= 0. \end{aligned}$$

Solution. Let x_0 be a common root of these equations. Then it is obvious that $x_0 \neq 0$. Then we have

$$\begin{aligned} x_0^3 + mx_0 + 1 &= 0, \\ x_0^4 + mx_0^2 + 1 &= 0. \end{aligned}$$

Multiplying both sides of the first equation by x_0 and subtracting the second equation from it, we get $x_0 = 1$. This means that if the given equations have a common root for some values of m, then this root must be equal to 1. Plug in $x = 1$ to either of the given equations, and get $m = -2$. In other words, in order for $x = 1$ to be a common root of the given equations, it is necessary that $m = -2$. The sufficiency of this condition follows from direct substitution $x = 1$ and $m = -2$ into the other equation.

Answer: -2

6. Given p and q are positive integers satisfying the following equality $p! + 12 = q^2$, find $p + q$. (Recall that $p! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (p-1) \cdot p$.)

Solution. First note that if $p \geq 5$ then $p!$ is proportional to 2 and to 5, i.e. $p!$ is divisible by 10 and thus the number $p! + 12$ has 2 as its last digit. But such a number cannot be a perfect square of an integer. It remains to consider cases $p = 1, 2, 3, 4$. It is easy to conclude that only the pair $(p, q) = (4, 6)$ satisfies the given equation. Finally, $p + q = 10$.

Answer: 10

7. Find the largest value a such that one of the roots of the following equation is greater than or equal to 1 and the other is less than or equal to 1:

$$(a^2 + a + 1)x^2 + (2a - 3)x + a - 5 = 0.$$

Solution. Consider the following function:

$$f(x) = (a^2 + a + 1)x^2 + (2a - 3)x + a - 5.$$

Note that $a^2 + a + 1 > 0$ for all real a. Thus the graph of $y = f(x)$ is a parabola which opens upward. Further, it follows from the problem that the point $x = 1$ lies between the x-intercepts of the graph of $y = f(x)$. Thus, in order for the roots of the equation (= the x-intercepts of the graph of $y = f(x)$) to satisfy the condition of the problem, it is necessary and sufficient that $f(1) \leq 0$. But $f(1) = a^2 + 4a - 7$. It remains to solve the inequality $a^2 + 4a - 7 \leq 0$ which yields $-2 - \sqrt{11} \leq a \leq -2 + \sqrt{11}$.

Answer: $-2 + \sqrt{11}$

8. The sides of a triangle satisfy the following inequalities:

$$a \leq 5 \leq b \leq 6 \leq c \leq 8.$$

Find the maximum possible area of such triangle.

Solution. Let γ be the angle between a and b. Then the area of triangle can be found by the formula: $A = \frac{1}{2}ab\sin\gamma$. Note that $\sin\gamma \leq 1$ and equality is attained when γ is right angle. Hence, $A \leq \frac{1}{2}ab \leq \frac{1}{2} \cdot 5 \cdot 6 = 15$, because the maximal values for a and b are 5 and 6, respectively. In this case the hypotenuse c is equal $\sqrt{61}$ and $6 < \sqrt{61} < 8$.

Answer: 15

9. A function $f(x)$ satisfies $2f(x) + f(x^2 - 1) = 1$ for all real x. Find $f(-\sqrt{2})$.

Solution. If $x = -\sqrt{2}$ we get $2f(-\sqrt{2}) + f(1) = 1$. To determine $f(1)$, consider the given equation when $x = 0, 1, -1$. We have

$$\begin{aligned} 2f(0) + f(-1) &= 1 \\ 2f(1) + f(0) &= 1 \\ 2f(-1) + f(0) &= 1. \end{aligned}$$

It follows $f(0) = f(1) = f(-1) = \frac{1}{3}$. Thus, $f(-\sqrt{2}) = 1/3$.

Answer: $\frac{1}{3}$

10. How many solutions does the equation

$$\sqrt{x + 6 - 2x^2} \cdot \cos(\pi x) = 0$$

have?

Solution. The given equation is equivalent to the following

$$x + 6 - 2x^2 = 0 \quad \text{or} \quad \left\{ \begin{array}{r} \cos(\pi x) = 0, \\ x + 6 - 2x^2 \geq 0, \end{array} \right.$$

The first equation has solutions $x = -\frac{3}{2}; 2$. The second system implies

$$\begin{array}{r} \pi x = \frac{\pi}{2} + \pi k, \\ 2x^2 - x - 6 \geq 0, \end{array}$$

where k is an arbitrary integer, or

$$\begin{array}{r} x = \frac{1}{2} + k, \\ -\frac{3}{2} \leq x \leq 2. \end{array}$$

So, the given equation has the following solutions $x = -\frac{3}{2}; -\frac{1}{2}; \frac{1}{2}; \frac{3}{2}; 2$.

Answer: 5

11. Compute $S = \cos 36^\circ - \sin 18^\circ$.

Solution.

$$S = \cos 36^\circ - \sin 18^\circ = \frac{\cos 36^\circ \cos 18^\circ - \sin 18^\circ \cos 18^\circ}{\cos 18^\circ}.$$

Using the known trigonometric identities,

$$\begin{aligned} \cos A \cos B &= \tfrac{1}{2}(\cos(A - B) + \cos(A + B)) \\ \sin A \cos A &= \tfrac{1}{2}\sin(2A) \end{aligned}$$

we get

$$S = \frac{\frac{1}{2}\cos 18^\circ + \frac{1}{2}\cos 54^\circ - \frac{1}{2}\sin 36^\circ}{\cos 18^\circ}.$$

Note that $\cos 54^\circ = \cos(90 - 36^\circ) = \sin 36^\circ$. Hence,

$$S = \frac{\frac{1}{2}\cos 18^\circ}{\cos 18^\circ} = \frac{1}{2}.$$

Answer: $\frac{1}{2}$

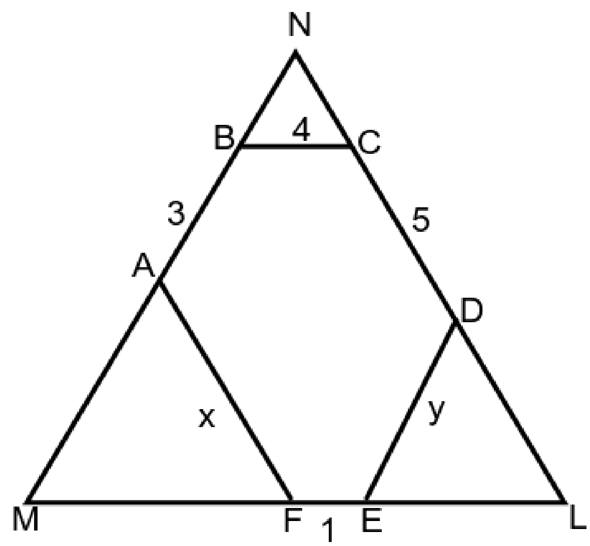
12. Given sequence $a_1 = 2$, $a_2 = 6$, $a_3 = 12$, $a_4 = 20$, $a_5 = 30$, $a_6 = 42, \ldots$. Compute a_{2013}.

    Solution. Note that $a_2 - a_1 = 2 \cdot 2$, $a_3 - a_2 = 2 \cdot 3$, $a_4 - a_3 = 2 \cdot 4$, $a_5 - a_4 = 2 \cdot 5$, $a_6 - a_5 = 2 \cdot 6, \ldots$ Thus, $a_{2013} - a_{2012} = 2 \cdot 2013$. So, $a_{2013} - a_1 = 2 \cdot 2 + 2 \cdot 3 + 2 \cdot 4 + \ldots + 2 \cdot 2013$, and $a_{2013} = 2 + 2 \cdot 2 + 2 \cdot 3 + 2 \cdot 4 + \ldots + 2 \cdot 2013 = 2 \cdot (1 + 2 + 3 + \ldots + 2013) = 2 \cdot 2013 \cdot 2014/2 = 4\ 054\ 182$.

    Answer: 4 054 182

13. Each angle of the hexagon $ABCDEF$ is $120°$. Find $AF + DE$ if $AB = 3$, $BC = 4$, $CD = 5$ and $EF = 1$.

    Solution. We extend the sides AB, CD and EF to their pairwise intersection of the points M, N and L (see the picture below).



    Then each of the triangles MAF, BNC and DLE contains two angles that are $60°$, so these triangles are equilateral. Consequently, the triangle MNL is also equilateral. Then $AF + 3 + 4 = 4 + 5 + DE = y + 1 + AF$, where $AF = 8$, $DE = 6$. Finally, $AF + DE = 14$.

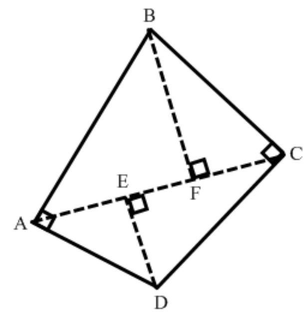
    Answer: 14

14. Let $\spadesuit(n)$ denote the sum of the digits of the positive integer n. For example, $\spadesuit(5) = 5$ and $\spadesuit(8123) = 8 + 1 + 2 + 3 = 14$. For how many two-digit values of n is $\spadesuit(\spadesuit(n)) = 3$?

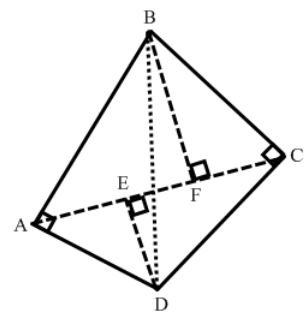
    Solution. Note that the maximum possible value for $\spadesuit(n)$ when n is a two-digit integer is $9 + 9 = 18$. Let $p = \spadesuit(n)$. In order to have $\spadesuit(p) = 3$, we must have p as 3,12, or 21. However $p = 21 > 18$ does not give valid solutions for n. If $3 = p = \spadesuit(n)$, then $n = 12, 21$, or 30, and if $12 = p = \spadesuit(n)$, then $n = 39, 93, 48, 84, 57, 75$, or 66. This gives 10 total possibilities for n.

    Answer: 10.

15. The quadrilateral $ABCD$ has right angles at A and at C. Points E and F are on AC with DE and BF perpendicular to AC. In addition, $AE = 3$, $DE = 5$, and $CE = 7$. What is BF?



Solution. First construct the diagonal BD.



Since $\angle BAD$ is a right angle, $\angle BAE + \angle EAD = 90°$. But the triangle AFB is a right triangle, so $\angle BAE + \angle ABF = 90°$ as well. Thus $\angle EAD = \angle ABF$, and the right triangle AED is similar to the right triangle AFB. Hence,

$$\frac{AF}{BF} = \frac{DE}{AE},$$

so

$$AF = \frac{DE}{AE}BF = \frac{5}{3}BF.$$

Similarly, $\angle CBF = \angle DCE$, and the right triangle CFB is similar to the right triangle DEC. So

$$\frac{CF}{BF} = \frac{DE}{EC},$$

so

$$CF = \frac{DE}{EC}BF = \frac{5}{7}BF.$$

Thus

$$AE + EC = 3 + 7 = 10 = AF + FC = \frac{5}{3}BF + \frac{5}{7}BF = \frac{50}{21}BF$$

and $BF = 21/5$.

Answer: 21/5

16. Three cubes having volumes 1, 8, and 27 are glued together at their faces. What is the smallest possible surface area that the resulting polyhedron can have?

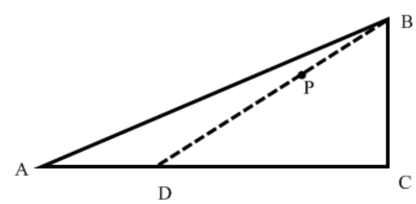
Solution. The total surface area of the three cubes before they have been placed together is

$$6 \cdot (3 \times 3 + 2 \times 2 + 1 \times 1) = 84.$$

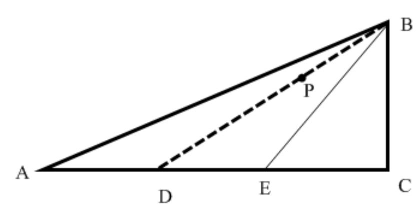
Placing the cube with volume 8 on one corner of the top face of the cube with volume 27 removes $2 \times 2 = 4$ square units of the surface area from each cube. This reduces the surface area to $84 - 2 \times 4 = 76$. Placing the unit cube so that one of its faces adjoins the cube with volume 27 and another adjoins the cube with volume 8 reduces the surface area an additional 4 square units, leaving the minimal surface area of $84-2\cdot4-4\cdot1=72$ square units.

Answer: 72

17. Triangle ABC is a right triangle with $\angle ACB$ as its right angle, $\angle ABC = 60°$, and $AB = 10$. Point P is randomly chosen inside the triangle ABC, and the segment BP is extended to meet the side AC at D. What is the probability that $BD > 5\sqrt{2}$?



Solution. Since $AB = 10$ and $\angle ABC = 60°$, we have $BC = 5$ and $AC = \sqrt{10^2 - 5^2} = 5\sqrt{3}$. Suppose that E is the point on the segment AC with $BE = 5\sqrt{2}$ (in other words, we consider the boundary case). Then $CE = \sqrt{(5\sqrt{2})^2 - 5^2} = 5$, and triangle ECB is an isosceles right triangle.



So $BD < 5\sqrt{2}$ when D is between E and C, that is, when P is inside the triangle BEC. The probability that this will occur is

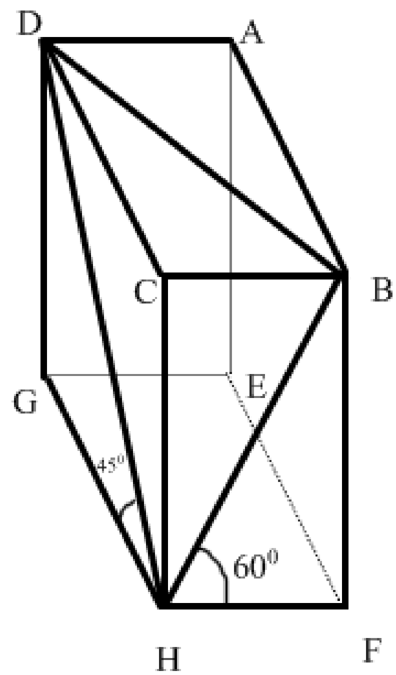
$$\frac{Area(\triangle BEC)}{Area(\triangle BAC)} = \frac{\frac{1}{2}5^2}{\frac{1}{2}5^2\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

Hence the probability that $BD > 5\sqrt{2}$ is

$$1 - \frac{\sqrt{3}}{3} = \frac{3-\sqrt{3}}{3}.$$

Answer: $\frac{3-\sqrt{3}}{3}$

18. In the rectangular solid shown, we have $\angle DHG = 45°$ and $\angle FHB = 60°$. What is the cosine of $\angle BHD$?



Solution. Since no dimensions have been specified, we can assign a value to one of them and determine the others relative to that dimension. Let $GH = 1$. Since $\angle GHD = 45°$, this implies that

$$1 = GH = DG = DC = CH = BF$$

and $DH = \sqrt{2}$. In addition, since $\triangle HFB$ is $30 - 60 - 90°$ with its longest leg $BF = 1$, we have

$$BH = BD = \frac{2\sqrt{3}}{3}$$

and

$$BC = HF = \frac{1}{2}BH = \frac{\sqrt{3}}{3}.$$

Applying the Law of Cosines to $\triangle BHD$ gives

$$BD^2 = DH^2 + BH^2 - 2 \cdot DH \cdot BH \cos \angle BHD,$$

so

$$\cos \angle BHD = \frac{1}{2 \cdot DH \cdot BH}(DH^2 + BH^2 - BD^2) = \frac{1}{2 \cdot \sqrt{2} \cdot \frac{2\sqrt{3}}{3}}((\sqrt{2})^2 + (\frac{2\sqrt{3}}{3})^2 - (\frac{2\sqrt{3}}{3})^2) = \frac{\sqrt{6}}{4}.$$

Answer: $\frac{\sqrt{6}}{4}$

19. Suppose that

$$\log_{2013}(\log_{2014}(\log_{2015}(\log_7 N))) = 2012.$$

How many different prime numbers are factors of N?

Solution.

$$\log_{2013}(\log_{2014}(\log_{2015}(\log_7 N))) = 2012$$

implies that

$$\log_{2014}(\log_{2015}(\log_7 N)) = 2013^{2012},$$

and also

$$\log_{2015}(\log_7 N) = 2014^{2013^{2012}},$$

which in its turn yields

$$\log_7 N = 2015^{2014^{2013^{2012}}}$$

and, finally that

$$N = 7^{2015^{2014^{2013^{2012}}}}.$$

In summary then, N has only the prime factor 7.

Answer: 1

20. Given that $\sin x = 3\cos x$. What is $\sin x \cos x$?

Solution. Since

$$1 = \sin^2 x + \cos^2 x = (3\cos x)^2 + \cos^2 x = 10\cos^2 x,$$

we have

$$\cos x = \pm\frac{\sqrt{10}}{10}$$

and

$$\sin x = 3\cos x = \pm\frac{3\sqrt{10}}{10}.$$

Since $\sin x = 3\cos x$, the sign chosen for the sine must match the sign for the cosine. Hence,

$$\sin x \cos x = \pm\frac{3\sqrt{10}}{10} \cdot \pm\frac{\sqrt{10}}{10} = \frac{3}{10}.$$

Answer: 3/10