EF Exam

Texas A&M Math Contest

16 November, 2013

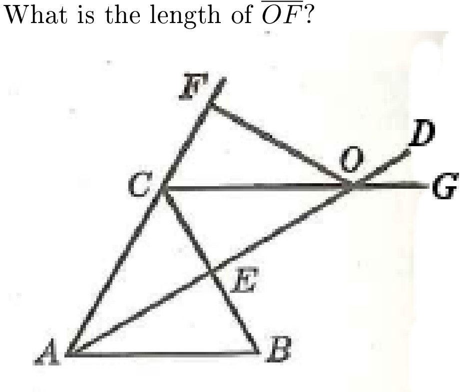
(NOTE: If units are appropriate, please include them in your answer. All answers must be simplied where possible.)

1. Paul shared his baseball cards with three friends. He gave half his cards to one friend, one-third of the cards that were left to a second friend, and the remaining 12 cards to a third friend. How many cards did Paul have to start with?

2. Determine the remainder when 2013^{2013} is divided by 10.

3. The product of two numbers is 12. The ratio of the difference of their cubes to the cube of their difference is 19. What is the sum of the numbers?

4. In the figure below, $AB = BC = CA = 6$, $\overline{AD}$ bisects $\angle A$, $\overline{CG}$ bisects $\angle BCF$, and $\overline{OF} \perp \overline{AC}$. What is the length of $\overline{OF}$?



5. Given quadrilateral $TAMU$ with $AU = 10$, $AM = 5$, $m\angle T = 30°$, $m\angle MUT = 60°$, and $m\angle TAU = m\angle AMU$. What is MU?

6. Find the polynomial whose roots are the reciprocals of the roots of $p(x) = 2x^2 - x - 13$. Give your answer in the form $q(x) = ax^2 + bx + c$ where a, b, and c are relatively prime integers (i.e., their greatest common factor is 1), and $a > 0$.

7. Two distinct complex numbers w and z have the property that each is the square of the other. What is $|w - z|$?

8. Determine the smallest positive solution to the equation $\tan(3\theta) = \cot(4\theta)$.

9. Evaluate $\tan(54°)(\cos(54°) + \cos(162°))$.

10. It can be shown that, if $y = mx + b$ is tangent to the graph of $f(x) = x^3$ at a nonzero value of x, then it will intersect the graph of f at another point. Find the slope of the line tangent to f at this intersection.

11. Given $\int_0^2 f(x)\,dx = 6$ and $\int_2^4 f(x)\,dx = 12$, what is $\int_0^2 f(2x)\,dx$?

12. Let f be a differentiable function such that $f(x+h) - f(x) = 3x^2h + 3xh^2 + h^3 + 2h$ for all x and h and $f(0) = 1$. If $g(x) = e^{-x}f(x)$, find $g'(3)$.

13. For $-1 < x < 1$ define $f(x) = \sum_{n=0}^{\infty}(-1)^{n(n-1)/2}x^n$. Write a rational function which is equivalent to f for $-1 < x < 1$.

14. Let $x_1 = 1$, and define $x_{n+1} = \sec(\arctan(x_n))$. What is x_{2013}?

15. Let C_1 be the circle $(x-1)^2 + y^2 = 1$ and C_2 be the circle with radius $\overline{OP}$ where O is the origin and P the point $(0,\, r)$ $(r \leq 2)$. Let Q be the upper point of intersection of C_1 and C_2 and the point $(a,\, 0)$ be the x-intercept of the line passing through P and Q. Find $\lim\limits_{r \to 0^+} a$.

16. Evaluate $\displaystyle\int_0^1 \frac{\ln(x+1)}{x^2+1}$.

17. Let f be a function whose domain and range is $(0,\, \infty)$ and a be a number such that $f'\left(\frac{a}{x}\right) = \frac{x}{f(x)}$ for all x. If $f(1) = 2$ and $f'(1) = 6$, what is $f(x)$?

18. Let a and b be real numbers greater than 1 for which there exists a positive real number c $(c \neq 1)$ such that

$$2(\log_a c - \log_b c) = 3\log_{ab} c$$

Find $\log_a b$.

19. Let k be an integer with $0 \leq k \leq 2013$. Define $f(k) = \binom{2013}{k}\left(\frac{1}{6}\right)^k\left(\frac{5}{6}\right)^{2013-k}$. Find the value of k which maximizes f.
RECALL: $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$