Solutions to EF Exam

Texas A&M High School Math Contest

16 November, 2013

1. Let n be the number of baseball cards Paul started with. He gave his third friend $n - \frac{1}{2}n - \frac{1}{3} \cdot \frac{1}{2}n = 12$ cards, so $6n - 3n - n = 72$, or $n = \mathbf{36}$ **cards**.

2. The remainder is the same as when 3^{2013} is divided by 10. The sequence of numbers $3^n|10$ is $\{3, 9, 7, 1, 3, 9\cdots\}$, so $2013^{2013}|10 = 3^{2013}|10 = 3^1|10 = \mathbf{3}$.

3. Let x and y be the numbers. Then $xy = 12$, and $\dfrac{x^3 - y^3}{(x-y)^3} = \dfrac{x^2 + xy + y^2}{x^2 - 2xy + y^2} = \dfrac{x^2 + 12 + y^2}{x^2 - 24 + y^2} = 19$. Clearing the fractions yields $18x^2 + 18y^2 = 12 + 19(24)$, or $x^2 + y^2 = 26$. Adding $2xy = 24$ to both sides yields $(x+y)^2 = 50$, or $x + y = \mathbf{5\sqrt{2}}$.

4. Since $\triangle ABC$ is an equilateral triangle, $\triangle ACE$ is a 30-60-90 triangle. Further, $m\angle BCF = 120°$, so $\triangle OCE$ and $\triangle OCF$ are also 30-60-90 triangles which are congruent to $\triangle ACE$. Therefore, $OF = AE = \mathbf{3\sqrt{3}}$.

5. Extend segments $\overline{TA}$ and $\overline{UM}$ until they intersect at point X. Since $\angle AMX = \angle UAX$, $\triangle AMX \sim \triangle UAX$, so $\dfrac{MX}{AX} = \dfrac{AM}{UA} = \dfrac{5}{10}$, or $AX = 2MX$. From right triangle AMX, $MX^2 + (2MX)^2 = 5^2$, or $MX = \sqrt{5}$ and $AX = 2\sqrt{5}$. Since $\dfrac{AX}{UX} = \dfrac{5}{10}$ as well, we have $UX = 2AX = 4\sqrt{5}$. Therefore, $UM = UX - MX = \mathbf{3\sqrt{5}}$.

6. Let r and s be the roots of $p(x)$. Then $r + s = -\dfrac{1}{2}$ and $rs = -\dfrac{13}{2}$. The sum and product of the roots of $q(x)$ must be $\dfrac{1}{r} + \dfrac{1}{s} = \dfrac{r+s}{rs} = \dfrac{1}{13}$ and $\dfrac{1}{r} \cdot \dfrac{1}{s} = -\dfrac{2}{13}$. Therefore, $q(x) = x^2 + \dfrac{1}{13}x - \dfrac{2}{13} = \mathbf{13x^2 + x - 2}$.

7. Let z and w be the two numbers. Since $z^2 = w$ and $w^2 = z$, we must have $z^4 = z$, or $z(z-1)(z^2+z+1) = 0$. If $z = 0$, $w = 0$ and the numbers are not distinct. Similarly, if $z = 1$, $w = 1$ and again the numbers are not distinct. Therefore, we must have $z^2 + z + 1 = 0$, or, from the quadratic formula: $z = \dfrac{-1 \pm \sqrt{1-4}}{2} = -\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}i$. But $w^4 = w$ as well, so we also have $w = -\dfrac{1}{2} \mp \dfrac{\sqrt{3}}{2}i$. Then $z - w = \sqrt{3}\,i$ or $-\sqrt{3}\,i$, so $|z - w| = \mathbf{\sqrt{3}}$.

8. $\tan(3\theta) = \cot(4\theta) = \tan\left(\dfrac{\pi}{2} - 4\theta\right)$. Thus, $3\theta = n\pi + \left(\dfrac{\pi}{2} - 4\theta\right)$, or $\theta = \dfrac{2n\pi + \pi}{14}$. Therefore, the smallest positive solution is $\theta = \dfrac{\pi}{\mathbf{14}}$.

9. $\tan(54°)(\cos(54°) + \cos(162°) = \sin(54°) + \dfrac{\sin(54°)\cos(162°)}{\cos(54°)} = \sin(54°) + \dfrac{\sin(216°) - \sin(108°)}{2\cos(54°)}$ $= \sin(54°) - \dfrac{\sin(36°)}{2\cos(54°)} - \dfrac{2\sin(54°)\cos(54°)}{2\cos(54°)} = -\dfrac{\cos(54°)}{2\cos(54°)} = -\dfrac{\mathbf{1}}{\mathbf{2}}$.

10. Let a be the x-coordinate of the point of tangency. Then $m = f'(a) = 3a^2$, and the equation of the tangent line is $y = a^3 + 3a^2(x - a)$. This line intersects $y = x^3$ when $a^3 + 3a^2x - 3a^3 = x^3$, or $x^3 - 3a^2x + 2a^2 = 0$. Using the fact that the line is tangent at $x = a$, we can factor the left as $(x-a)^2(x+2a) = 0$, so the line intersect f at $x = -2a$. Therefore, the slope of the tangent line at the point is $3(-2a)^2 = 12a^2 = \mathbf{4m}$.

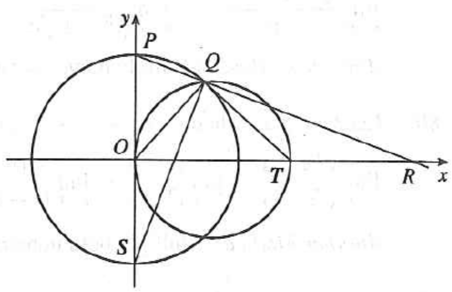
11. By letting $u = 2x$, we have $\int_0^2 f(2x)\,dx = \frac{1}{2}\int_0^4 f(u)\,du = \frac{1}{2}(6+12) = \mathbf{9}$.

12. $g'(x) = e^{-x}f'(x) - e^{-x}f(x)$. $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} 3x^2+3xh+h^2+2 = 3x^2+2$ and $f(x) = x^3+2x+f(0) = x^3+2x+1$. Therefore, $g'(3) = e^{-3}(29) - e^{-3}(34) = \mathbf{-5e^{-3}}$.

13. $f(x) = 1+x-x^2-x^3+x^4+x^5-x^6-x^7+\cdots = (1-x^2+x^4-x^6+x^8\cdots)+(x-x^3+x^5-x^7+x^9+\cdots)$. Since the two expressions are convergent geometric series, their sum is $\frac{1}{1+x^2}+\frac{x}{1+x^2} = \mathbf{\frac{1+x}{1+x^2}}$.

14. For $x > 0$, $\sec(\arctan(x)) = \sqrt{\tan^2(\arctan(x))+1} = \sqrt{x^2+1}$, so $x_2 = \sqrt{2}$, $x_3 = \sqrt{3}$, $\cdots x_{2013} = \sqrt{\mathbf{2013}}$.

15. Using algebra and calculus, we can show that C_1 and C_2 intersect in the point $Q\left(\frac{r^2}{2}, \frac{r}{2}\sqrt{4-r^2}\right)$ and the equation of the line through P and Q is $y = \frac{\sqrt{4-r^2}-2}{r}x+r$. Therefore, the x-intercept is $a = -\frac{r^2}{\sqrt{4-r^2}-2} = \sqrt{4-r^2}+2$, so $\lim_{r\to 0^+} a = \mathbf{4}$.
Alternately, we can find the limit geometrically using the figure below, defining R to be the desired x-intercept :



$\angle PQS$, $\angle OQT$, and $\angle SQR$ are all right angles (first 2 subtend diameters; the last is complementary to the first), so $\angle OQS = \angle TQR$. Further, using appropriate right triangles we see that $\angle OSQ = \angle TRQ = 90 - \angle OPQ$, so $\triangle SOQ \sim \triangle RTQ$. But $\triangle SOQ$ is isosceles; therefore $\triangle RTQ$ is also isosceles and $QT = TR$. As $r \to 0^+$, point Q approaches the origin, so $QT \to OT = 2$ and $TR \to 2$. Therefore, $OR \to 4$, so $a \to \mathbf{4}$.

16. Define $f(t) = \int_0^1 \frac{\ln(tx+1)}{x^2+1}\,dx$. Then $f(0) = 0$, $f(1)$ =our desired solution, and $f'(t) = \int_0^1 \frac{x}{(tx+1)(x^2+1)}\,dx$. Using partial fractions, we obtain $f'(t) = \frac{1}{t^2+1}\int_0^1\left(\frac{-t}{tx+1}+\frac{x+t}{x^2+1}\right)dx$ $= \frac{1}{t^2+1}(-\ln(tx+1)+\frac{1}{2}\ln(x^2+1)+t\arctan x|_{x=0}^{x=1} = \frac{1}{t^2+1}(-\ln(t+1)+\frac{1}{2}\ln 2+\frac{\pi t}{4}$. Integrate both sides from 0 to t: $f(t)-f(0) = \frac{1}{2}\ln 2\arctan t+\frac{\pi}{8}\ln(t^2+1) - \int_0^t \frac{\ln(x+1)}{x^2+1}\,dx$. Since the last term on the right is $f(t)$, we have $f(t) = \frac{1}{4}\ln 2\arctan t+\frac{\pi}{16}\ln(t^2+1)$. Therefore, $f(1) = \mathbf{\frac{\pi\ln 2}{8}}$

17. Since $f'\left(\dfrac{a}{x}\right) = \dfrac{x}{f(x)}$, we also have $f'(x) = f'\left(\dfrac{a}{\frac{a}{x}}\right) = \dfrac{\frac{a}{x}}{f\left(\frac{a}{x}\right)} = \dfrac{a}{xf\left(\dfrac{a}{x}\right)}$. Differentiate both sides to yield $f''(x) = -\dfrac{a}{x^2 f\left(\dfrac{a}{x}\right)} + \dfrac{a^2 f'\left(\dfrac{a}{x}\right)}{x^3\left(f\left(\frac{a}{x}\right)\right)^2}$. Use the first two equations to substitute for $f\left(\dfrac{a}{x}\right)$ and $f'\left(\dfrac{a}{x}\right)$ to yield $f''(x) = -\dfrac{f'(x)}{x} + \dfrac{(f'(x))^2}{f(x)}$. Clear the fractions and the right hand side to obtain $xf(x)f''(x) + f'(x)f(x) - x(f'(x))^2 = 0$. Divide by $(f(x))^2$: $\dfrac{xf(x)f''(x) + f'(x)f(x) - x(f'(x))^2}{(f(x))^2} = 0$, or $\dfrac{f(x)(xf''(x) + f'(x)) - f'(x)\cdot xf'(x)}{(f(x))^2} = 0$**.** Since the left side is the derivative of $\dfrac{xf'(x)}{f(x)}$, this expression must be a constant (call it d). Then $\dfrac{f'(x)}{f(x)} = \dfrac{d}{x}$. Integrate both sides to obtain $\ln(f(x)) = d\ln x + C$, or $f(x) = cx^d$**.** $f(1) = 2$ implies that $c = 2$, and $f'(1) = cd(1)^{d-1} = 6$ implies $d = 3$. Therefore, $f(x) = \mathbf{2x^3}$**.**

18. Let $x = \log_c a$ and $y = \log_c b$. Then $2\left(\dfrac{1}{x} - \dfrac{1}{y}\right) = \dfrac{3}{x+y}$. Clearing the fractions yields $2y(x+y) - 2x(x+y) = 3xy$, $2y^2 - 3xy - 2x^2 = 0$ $(2y+x)(y-2x) = 0$. Therefore, we have $2\log_c b = -\log_c a$ (not possible since $a > 1$ and $b > 1$) or $\log_c b = 2\log_c a$. Therefore, $\log_a b = \dfrac{\log_c b}{\log_c a} = \mathbf{2}$

19. If k is the location of the maximum value of f, then we must have $f(k) \geq f(k+1)$ and $f(k) \geq f(k-1)$, or (since $f(k) > 0$ for all k) $\dfrac{f(k)}{f(k+1)} \geq 1$ and $\dfrac{f(k)}{f(k-1)} \geq 1$. The first inequality yields $\dfrac{\binom{2013}{k}\left(\dfrac{1}{6}\right)^k\left(\dfrac{5}{6}\right)^{2013-k}}{\binom{2013}{k+1}\left(\dfrac{1}{6}\right)^{k+1}\left(\dfrac{5}{6}\right)^{2012-k}} \geq 1$, which simplifies to $\dfrac{5(k+1)}{2013-k} \geq 1$ which is true when $k \geq \dfrac{2008}{6} = 334\dfrac{2}{3}$. In like manner, the second inequality simplies to $\dfrac{2014-k}{5k} \geq 1$, which is true when $k \leq \dfrac{2014}{6} = 335\dfrac{2}{3}$. Therefore, f is maximized when $k = \mathbf{335}$.