

Best Student Closed Exam
Texas A&M High School Math Contest
November 16, 2013

1. How many zeros are there if you write out in full the number

$$N = (1000000)^{1000^{10}}?$$

- 2.

$$\frac{(x^5 - 1)(x^7 - 1)}{(x - 1)^2}$$

can be written in the form $a_0 + a_1x + a_2x^2 + \cdots + a_{10}x^{10}$. Find $a_0 + a_1 + \cdots + a_{10}$.

3. How many real numbers x are there such that $0 \leq x \leq 2\pi$ and $\sin x + \sin(2x) = \sin(3x) + \sin(4x)$?
4. Let N be the number of 13-card hands, drawn from a standard deck of 13 clubs, diamonds, hearts, and spades, that contain exactly 10 spades. Find the prime factorization of N .
5. How many real numbers x are there such that the fourth derivative of e^{-x^2} is zero at x ?
6. A frog takes two random hops, each to some random point one meter away from where it was when it started the hop. Its first hop thus carries it to some point on a circle about its initial position, and its second hop carries it to some point on a circle about where it first landed. What is the probability that the frog lands less than 1 meter from its initial position?

7. Find real numbers a and b so that

$$\frac{20}{1+i} + \frac{50}{2-i} = a + ib.$$

8. Let u , v , and w be the zeros of the polynomial $x^3 - 4x^2 + 3x + 1$. Find (the exact, integer) value of $u^2 + v^2 + w^2$.

The array of numbers below is to be used for problems 9, 10, and 11. Note that, as in the spirit of the Fibonacci triangle, there is a simple rule which generates it.

1	0	0	0	0	0	0
1	1	0	0	0	0	0
1	3	1	0	0	0	0
1	7	6	1	0	0	0
1	15	25	10	1	0	0
1	31	90	65	15	1	0
1	63	301	350	\ddots	\ddots	\ddots

9. Find the number just to the right of 350.
10. Say $A_{j,k}$ denotes the number in row j , column k , counting from 0, so that the '25' is $A_{4,2}$. Find a formula in terms of n for $A_{n,1}$.
11. There is a formula for $A_{n,2}$ along the same lines as the formula for $A_{n,1}$. Find that number C so that $A_{n,2}$ grows like C^n .

12. Find the radius r of the circle inscribed in a right triangle (so that the circle is tangent to each edge of the triangle) in terms of x and y , where the edges have lengths of x , y , and 1, with $0 < x < y < 1$. In particular, find r when $x = 3/5$ and $y = 4/5$.

13. Solve

$$\begin{aligned}xyz &= 4 \\x^2y^3z^4 &= 8 \\x^3y^4z^6 &= 16.\end{aligned}$$

14. Find

$$2^{32} \pmod{97}.$$

(Equivalently, find the remainder R in the long division calculation $2^{32} = 97Q + R$ with integers Q and R , where $0 \leq R \leq 96$.)

15. Simplify

$$\frac{x^4 + x^2 + 1}{x^2 + x + 1}.$$

16. Let

$$f(x) = \frac{1}{1 + \frac{1}{1 + \frac{1}{1+x}}}.$$

Find $f'(x)$ evaluated at $x = 1$ and fully simplified.

17. Exactly how many lattice points (points with integer coordinates) are there inside the triangle with vertices $(0, 0)$, $(100, 0)$, and $(100, 47)$? (Points on the edges or on the corners don't count as 'inside'.)
18. Consider a die with two red faces and four green faces. Let P be the probability that five tosses will yield four greens and one red. Let Q be the probability that nine tosses will yield six greens and three reds. Find P/Q .

Problems 19 and 20 refer to the following material.

A *permutation* of $\{1, 2, \dots, n\}$ is a one-to-one function from $\{1, 2, \dots, n\}$ to itself. Here, we write permutations by listing their values at $1, 2, \dots, n$, so that $(2, 3, 1)$ denotes the permutation that takes 1 to 2, 2 to 3, and 3 to 1, and (213) represents the permutation that takes 1 to 2, 2 to 1, and 3 to 3.

The composition $p \circ q$ of two permutations p and q is the permutation r so that $r(k) = p(q(k))$ for $1 \leq k \leq n$. The *period* of a permutation is the least positive integer k so that $p \circ p \circ p \cdots$ (k times) equals the identity permutation that takes every number to itself. (Thus, the period of $(2, 3, 1)$ is three.)

19. Find the period of $(2, 4, 5, 1, 3)$.
20. Find the expected value of the period of a random permutation on $\{1, 2, 3, 4\}$.