Solutions Best Student Exams

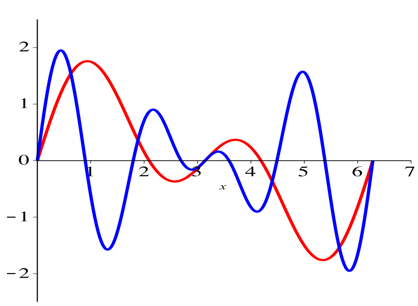
Texas A&M High School Math Contest
November 16, 2013

1. How many zeros are there if you write out in full the number
$$N = (1000000)^{1000^{10}}?$$
$N = 10^{(6\cdot 1000^{10})} = 10^{(6\cdot 10^{30})}$ so there are $6 * 10^{30}$ or
$$6000000000000000000000000000000$$
zeros in this number.

2.
$$\frac{(x^5-1)(x^7-1)}{(x-1)^2}$$
can be written in the form $a_0 + a_1x + a_2x^2 + \cdots + a_{10}x^{10}$. Find $a_0 + a_1 + \cdots + a_{10}$. Begin with $(x^n - 1)/(x-1) = 1 + x + \cdots + x^{n-1}$. Using this with $n = 5$ and $n = 7$ and *only then* setting $x = 1$ yields 35 as the answer.

3. How many real numbers x are there such that $0 \le x \le 2\pi$ and $\sin x + \sin(2x) = \sin(3x) + \sin(4x)$? The key here has to be to get $\sin nx$ all on the 'same page', using the trig identities $\sin(a+b) = \sin a \cos b + \cos a \sin b$ and $\cos(a+b) = \cos a \cos b - \sin a \sin b$. So
$$\sin 2x = 2\sin x \cos x$$
$$\sin 3x = \sin x \cos 2x + \cos x \sin 2x = \sin x(4\cos^2 x - 1)$$
$$\sin 4x = 2\sin 2x \cos 2x = 4\sin x \cos x(2\cos^2 x - 1) = \sin x(8\cos^3 x - 4\cos x).$$
Thus
$$\sin x + \sin 2x - \sin 3x - \sin 4x = -2\sin x(4\cos^3 x + 2\cos^2 x - 3\cos x - 1).$$
Thus the x we seek are either numbers at which $\sin x = 0$ (these are 0, π, and 2π), or numbers whose cosine is a root of $4u^3 + 2u^2 - 3u - 1$. This polynomial is zero at $u = -1$ but with a positive derivative, negative at $u = 0$, and positive at $u = 1$, so it has zeros at $u_1 = -1$, at some point u_2 between -1 and 0, and at some point u_3 between 0 and 1. For the last two of these, there are two values of x with $\cos x = u_j$, one positive and one negative, while for $u_1 = -1$, there is just one value, $x = \pi$. Since that one's already been counted, we have in total 3 zeros for the first factor $\sin x$, and four *new* zeros for the other factor, for a grand total of 7. The answer is 7.

Here's a plot of the graphs of $\sin x + \sin 2x$ and $\sin 3x + \sin 4x$:



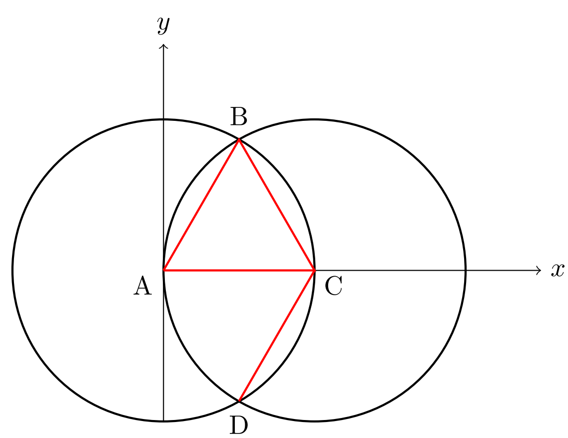
4. Let N be the number of 13-card hands, drawn from a standard deck of 13 clubs, diamonds, hearts, and spades, that contain exactly 10 spades. Find the prime factorization of N.

   There are
$$\binom{13}{10} = \binom{13}{3} = \frac{13 \cdot 12 \cdot 11}{1 \cdot 2 \cdot 3} = 13 \cdot 2 \cdot 11$$
   ways to choose which ten spades go in the hand. There are $\binom{39}{3} = 13 \cdot 19 \cdot 37$ ways to choose the other 3 cards. Multiplying these numbers yields $2 \cdot 11 \cdot 13^2 \cdot 19 \cdot 37 = N$.

5. How many real numbers x are there such that the fourth derivative of e^{-x^2} is zero at x? There are four. Every time you take another derivative, you introduce (at least) one new bump in the graph. (Any positive, infinitely differentiable function on $\mathbb{R}$ that, with all of its derivatives, tends to zero as x tends to $\pm\infty$, has at least one place where the first derivative is zero, and then on either side of that, at least one place where the next derivative is zero, and so on). On the other hand, the fourth derivative will be a polynomial of degree 4 in x, times e^{-x^2}, so it can't have any more than four zeros. The answer is 4.

6. A frog takes two random hops, each to some random point one meter away from where it was when it started the hop. Its first hop thus carries it to some point on a circle about its initial position, and its second hop carries it to some point on a circle about where it first landed. What is the probability that the frog lands less than 1 meter from its initial position?

   By symmetry, we may as well set up a coordinate system and call the positive x axis the direction of the first hop. Thus, the frog starts its second hop at $(1, 0)$. Now it can arrive from there at any point of the form $(1 + \cos x, \sin x)$ on the circle of radius 1 about $(1, 0)$. This circle intersects the circle of radius 1 about $(0, 0)$ at $(1/2, \pm\frac{1}{2}\sqrt{3})$. The probability that the frog lands inside the first circle is the ratio of that part of the second circle, which lies inside the first circle to the circumference of the second circle.



   Triangle ABC in the figure above is an equilateral triangle, which means that angle ACB is 60 degrees, which means that angle BCD is 120 degrees or $2\pi/3$ radians. Thus, arc BAD has length $2\pi/3$. Since the circumference of each circle is 2π, the desired ratio is $1/3$.
   The answer is $1/3$.

7. Find real numbers a and b so that
$$\frac{20}{1+i} + \frac{50}{2-i} = a + ib.$$
The basic idea is to 'rationalize the denominator' by writing $1/(u+iv) = (u-iv)/(u^2+v^2)$. That turns the original expression into $10 - 10i + 20 + 10i = 30 + 0i$ so $a = 30$ and $b = 0$.

8. Let u, v, and w be the zeros of the polynomial $x^3 - 4x^2 + 3x + 1$. Find the (exact, integer) value of $u^2 + v^2 + w^2$.

Begin by noting that
$$x^3 - 4x^2 + 3x + 1 = (x-u)(x-v)(x-w) = x^3 - (u+v+w)x^2 + (uv+uw+vw)x - uvw.$$
With polynomials, that means that the coefficients are equal on both sides. So $u+v+w = 4$, $uv+uw+vw = 3$, (and $uvw = -1$, though this last turns out not to be needed.) From this, it follows that $16 = (u+v+w)^2 = u^2+v^2+w^2+2(uv+uw+vw) = u^2+v^2+w^2+6$. Thus $u^2+v^2+w^2 = 10$. The answer is 10.

The array of numbers below is to be used for problems 9, 10, and 11. Note that, as in the spirit of the Fibonacci triangle, there is a simple rule which generates it.

$$\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 7 & 6 & 1 & 0 & 0 & 0 \\ 1 & 15 & 25 & 10 & 1 & 0 & 0 \\ 1 & 31 & 90 & 65 & 15 & 1 & 0 \\ 1 & 63 & 301 & 350 & \ddots & \ddots & \ddots \end{array}.$$

9. Find the number to the right of 350. The rule is one more than the column number times entry above, plus entry above left. That makes $65 + 5 \cdot 15 = 140$ as the answer.

10. Say $A_{j,k}$ denotes the number in row j, column k, counting from 0, so that the '25' is $A_{4,2}$. Find a formula in terms of n for $A_{n,1}$. The formula is $2^n - 1$. Proof by induction.

11. There is a formula for $A_{n,2}$ along the same lines as the formula for $A_{n,1}$. Find that number C so that $A_{n,2}$ grows like a constant times C^n. $C = 3$. There are two ways to see this. One is that if we start with the first 1 in column 2, its contribution to the entry $A_{n,2}$ is 3^{n-2}. The contribution from the '3' to its left is $3 \cdot 3^{n-3}$, and from the '7' we get $7 \cdot 3^{n-4}$, and so on. But the total of these other contributions is less than $\sum_{k=1}^{\infty} 2^k 3^{n-k} = 2 \cdot 3^n$. So our grand total is comparable to 3^n and $C = 3$.

    The other way is to guess that there is a formula for $A_{n,2}$ of the form $p \cdot 3^n + q \cdot 2^n + r$, work out from the starting entries $(0, 0, 1)$ of column 2 what p, q, r must be if the formula is to fit even the first few facts, and then prove by induction that it continues to work. As it happens, $p = 1/2$, $q = -1$, and $r = 1/2$ and the induction works.

12. Find the radius r of the circle inscribed in a right triangle (so that the circle is tangent to each edge of the triangle) with edges $0 < x < y < 1$, in terms of x and y. In particular, find r when $x = 3/5$ and $y = 4/5$.

    The answer is $xy/(1+x+y)$, which works out to $1/5$ when $x = 3/5$ and $y = 4/5$.

    With a coordinate system, we're looking at the triangle with vertices $(0,0), (-x,0)$, and $(0,y)$, and we want the distance from $(-r,r)$ to the line through $(-x,0)$ and $(0,y)$ to be r. Now the formula for this line had better be given in variables other than x and y so we can keep things straight—let's call them s and t. It will have the form $As + Bt = C$ and using the given points, that works out to $-ys + xt = xy$. The distance from a point (u,v) to the line $As + Bt = C$ is got by computing $D = Au + Bv$, and then it's $|C - D|/\sqrt{A^2+B^2}$. Here, $A = -y$ and $B = x$ and $C = xy$, so $A^2 + B^2 = 1$ and $|C - D| = |xy - (x+y)r|$. We want a point $(-r,r)$ inside our triangle so we need $xy > (x+y)r$ and so $xy - r(x+y) = r$. That gives $r = xy/(1+x+y)$ when the smoke clears.

13. Solve

$$\begin{aligned} xyz &= 4 \\ x^2y^3z^4 &= 8 \\ x^3y^4z^6 &= 16. \end{aligned}$$

Taking logs base 2, this becomes a linear problem. Let $u = \log x$, $v = \log y$, and $w = \log z$. Then we're solving $u + v + w = 2$, $2u + 3v + 4w = 3$, and $3u + 4v + 6w = 4$. The solution to this system is $u = 2$, $v = 1$, and $w = -1$, which translates to $x = 4$, $y = 2$, and $z = 1/2$.

14. Find

$$2^{32} \pmod{97}.$$

(Equivalently, find the remainder R in the long division calculation $2^{32} = 97Q + R$ with integers Q and R, where $0 \leq R \leq 96$.)

Two numbers are congruent mod c if their difference is a multiple of c, so for instance $256 \equiv 62 \pmod{97}$. The key fact here is that if $a \equiv A$ and $b \equiv B \pmod{c}$, then $AB \equiv ab \pmod{c}$.

That is, you can 'mod out' as you go. Now $2^4 = 16$ so $2^8 = 256 \equiv 256 - 2 \cdot 97(\mod 97) = 62$, so $2^{16} \equiv 62^2 = 3844$, and $3844 - 38 \cdot 97 = 158 \equiv 61$, so $2^{32} \equiv 61^2 = 3721 \equiv 35$. The answer is 35. And while we're at it, this trick is at the heart of the best known methods for testing whether a huge number is prime or not. If p is prime, then $2^{p-1} \equiv 1 \pmod{p}$. (A theorem due to Fermat). One need never actually compute 2^{p-1}, which would be insanely huge if p was huge already. Modding out as you go, and squaring so as to avoid having to compute all p different powers along the way, makes the computation feasible.

15. Simplify

$$\frac{x^4 + x^2 + 1}{x^2 + x + 1}.$$

The easiest way to do this is to just do the division. (Polynomial division works very much like integer long division).

The answer is $x^2 - x + 1$.

16. Let

$$f(x) = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{1+x}}}}.$$

Find $f'(x)$ evaluated at $x = 1$ and fully simplified.

It is best to simplify the fraction before taking the derivative. In general, $1 + \frac{a}{b} = \frac{a+b}{b}$ so the reciprocal of that is $\frac{b}{a+b}$. Applying this from the inside out yields $\frac{1}{1+x}$, $\frac{1+x}{2+x}$, $\frac{2+x}{3+2x}$, and finally $\frac{3+2x}{5+3x} = f(x)$. Now $f'(x) = (2(5 + 3x) - 3(3 + 2x))/(5 + 3x)^2 = 1/(5 + 3x)^2$. Setting $x = 1$ yields a derivative of $1/64$.

17. Exactly how many lattice points (points with integer coordinates) are there inside the triangle with vertices $(0, 0)$, $(100, 0)$, and $(100, 47)$? (Points on the edges or on the corners don't count as 'inside'.)

If we make another triangle with vertices $(0, 0)$, $(0, 47)$, and $(100, 47)$, we get a 100 by 47 rectangle with $99 \cdot 46$ strictly-interior points. Now none of these lie on the diagonal, because 100 and 47 are relatively prime. So exactly half of them belong inside our chosen triangle. That's 2277 points. The answer is 2277.

18. Consider a die with two red faces and four green faces. Let P be the probability that five tosses will yield four greens and one red. Let Q be the probability that nine tosses will yield six greens and three reds. Find P/Q.

We have $P = \binom{5}{1}(2/3)^4(1/3)^1 = 5 \cdot 2^4/3^5$, while $Q = \binom{9}{3}(2/3)^6(1/3)^3 = \frac{9 \cdot 8 \cdot 7}{6} \cdot \frac{2^6}{3^9}$. So

$$\frac{P}{Q} = \frac{5 \cdot 2^4 \cdot 6 \cdot 3^9}{3^5 \cdot 9 \cdot 8 \cdot 7 \cdot 2^6}.$$

This boils down to 135/112 or $5 \cdot 3^3/(7 \cdot 2^4)$.

Problems 19 and 20 refer to the following material.

A *permutation* of $(1, 2, \ldots, n)$ is a one-to-one function from $\{1, 2, \ldots, n\}$ to itself. Here, we write permutations by listing their values at $1, 2, \ldots, n$, so that $(2, 3, 1)$ denotes the permutation that takes 1 to 2, 2 to 3, and 3 to 1.

The composition $p \circ q$ of two permutations p and q is the permutation r so that $r(k) = p(q(k))$ for $1 \leq k \leq n$. The *period* of a permutation is the least positive integer k so that $p \circ p \circ p \cdots$ (k times) equals the identity permutation that takes every number to itself. (Thus, the period of $(2, 3, 1)$ is three.)

19. Find the period of $(2, 4, 5, 1, 3)$. It takes 3 moves to cycle 1 to 2 to 4 to 1, and it takes 2 moves to cycle 3 to 5 to 3. To synchronize those, we need six moves. So the answer is 6.

20. Find the expected value of the period of a random permutation on $\{1, 2, 3, 4\}$. There is one permutation with period 1: (1,2,3,4). There are nine permutations with period 2: six where two elements remain fixed and three with two pairings. There are eight permutations with period 3: four choices of fixed point times two choices for the cycle direction. The longest possible period here is 4 so that leaves 6 permutations with period 4. Thus the expected value is

$$\frac{1 \cdot 1 + 9 \cdot 2 + 8 \cdot 3 + 6 \cdot 4}{24} = \frac{67}{24}.$$

The answer is $\frac{67}{24}$.