2015 AB Exam

Texas A&M High School Math Contest
October 24, 2015

1. For how many integers n in the set $\{1, 2, 3, ..., 150\}$ is $n^3 - n^2$ the square of an integer?

2. If $\frac{1}{x} + \frac{1}{y} = 5$ and $\frac{1}{x} - \frac{1}{y} = 1$, find $x + y$.

3. Daisy has a bag containing 6 distinct objects. She draws one object and replaces it. She does this four times. What is the probability that she draws the same object exactly three times.

4. The sum of the first ten terms of a nonzero geometric series is 244 times the sum of the first five terms. What is the common ratio of this series?

5. Write 2015 in the base 5 numeration system.

6. How many members of the set $\{\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, ..., \frac{100}{7}\}$ are integer multiples of $\frac{2}{5}$?

7. If f is a function such that $f(1) = 1$ and $f(n) = f(n-1) + 2n - 1$ for all $n \geq 2$, find $f(100)$.

8. Find the value of the number A such that the three solutions to $x^3 - 8x^2 + Ax - 12 = 0$ are positive integers.

9. Let $P(x)$ be a monic polynomial of degree 3. (So the coefficient of x^3 is 1.) Suppose that the remainder when $P(x)$ is divided by $x^2 - 5x + 6$ equals 2 times the remainder when $P(x)$ is divided by $x^2 - 5x + 4$. If $P(0) = 100$, find $P(5)$.

10. How many positive integers less than 1400 have no repeating digits, i.e. no digit occurs more than once. (For example 103 is such an integer, but 131 is not.)

11. Find all possible ordered pairs (A, B) of digits such that the integer $7A8B$ is divisible by 45.

12. Among all real numbers x and y such that $|x + y| + |x - y| = 10$, what is the largest value of $x^2 + y^2 - 8y$?

13. How many ways can one list $1, 2, 3, 4$ so that no integer is followed by its successor, i.e. $n, n+1$ never occurs? For example $2, 1, 4, 3$ is one such listing but $2, 1, 3, 4$ is not (4 is the successor of 3).

14. Exactly one of the five integers listed below is a prime. Which one is the prime number?

    (a) $999,991$ (b) $999,973$ (c) $999,983$ (d) $1,000,001$ (e) $7,999,973$

15. The set of integers $\{1, 2, ..., 20\}$ is divided into k disjoint subsets $S_1, S_2, ..., S_k$ such that no two different integers whose sum is divisible by 5 are in the same subset. For example 7 and 8 are not in the same subset since 5 divides $7 + 8$. What is the least possible value of k?

16. A number system with base 26 uses the following symbols for the digits 0 through 25: $A = 0, B = 1, C = 2, ..., Y = 24, Z = 25$. Express $P + Q$ in this system.

17. How many ordered pairs (x, y) of integers satisfy $x^2 + 6x + y^2 = 16$?

18. Recall that if S is a set then $|S|$ denotes the number of elements in S. Now let $S = \{1, 2, 3, 4, 5, 6\}$. How many different nonempty subsets T of S do not contain $|T|$? For example $T = \{3, 5\}$ is one such subset since $|T| = 2$ is not in T.

19. Find all positive integers n of the form $n = p^2q$ with p and q distinct primes and such that the sum of the reciprocals of all of the divisors of n is 2.

20. In Dr. Stecher's special topics class, 20% of the students are juniors and 80% are seniors. On a recent test the average score for the entire class was 85 and the average score for the seniors was 88. What was the average score for the juniors?

21. Let f_1 be the function defined by $f_1(x) = 1 - \frac{1}{x}$. Let f_2 be defined by $f_2(x) = f_1(f_1(x))$. Let f_3 be defined by $f_3(x) = f_1(f_2(x))$. Continue in this fashion to define the sequence of functions $f_1, f_2, ..., f_n, ...$ Find the value of $f_{50}(50)$.